

Coherent pairs of measures and Markov-Bernstein inequalities

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Abstract

All the coherent pairs of measures associated to linear functionals c_0 and c_1 , introduced by Iserles et al in 1991, have been given by Meijer in 1997. There exist seven kinds of coherent pairs. All these cases are explored in order to give three term recurrence relations satisfied by polynomials. The smallest zero $\mu_{1,n}$ of each of them of degree n has a link with the Markov-Bernstein constant M_n appearing in the following Markov-Bernstein inequalities:

$$c_1((p')^2) \leq M_n^2 c_0(p^2), \quad \forall p \in \mathcal{P}_n,$$

where $M_n = \frac{1}{\sqrt{\mu_{1,n}}}$.

The seven kinds of three term recurrence relations are given. In the case where $c_0 = e^{-x}dx + \delta(0)$ and $c_1 = e^{-x}dx$, explicit upper and lower bounds are given for $\mu_{1,n}$, and the asymptotic behavior of the corresponding Markov-Bernstein constant is stated. Except in a part of one case, $\lim_{n \rightarrow \infty} \mu_{1,n} = 0$ is proved in all the cases.

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1 Introduction

The Markov-Bernstein inequalities in L^2 norm that imply two measures, have the following form

$$c_1((p')^2) \leq M_n^2 c_0(p^2), \quad \forall p \in \mathcal{P}_n,$$

where c_0 and c_1 denote the two linear functionals associated to the both measures. \mathcal{P}_n is the vector space of real polynomials in one variable of degree at most n . M_n is called Markov-Bernstein constant. These inequalities are always related to an eigenvalue problem of a positive definite symmetric matrix (see [13], [4] for a general presentation). For any measures this matrix generally is full. But for classical measures (Hermite, Laguerre-Sonin, Jacobi) this matrix has a particular

form. It is diagonal for all these measures (see [3]) if $c_0 = c_1 = \int_{-\infty}^{+\infty} .e^{-x^2} dx$ in the Hermite case, if $c_0 = c^\alpha = \int_0^{+\infty} .x^\alpha e^{-x} dx$ with $\alpha > -1$ and $c_1 = c^{\alpha+1}$ in the Laguerre-Sonin case, if $c_0 = c^{(\alpha,\beta)} = \int_{-1}^{+1} .(1-x)^\alpha (1+x)^\beta dx$ with $\alpha > -1$ and $\beta > -1$, and $c_1 = c^{(\alpha+1,\beta+1)}$ in the Jacobi case. It is tridiagonal (see [13], [4]) if $c_0 = c_1 = c^\alpha$ in the Laguerre-Sonin case. If $c_0 = c_1 = c^{(\alpha,\alpha)}$ that is the Gegenbauer case, two tridiagonal matrices are obtained (see [13], [4]). It is a five-diagonal matrix if $c_0 = c_1 = c^{(\alpha,\beta)}$ in the Jacobi case (see [4], [7]). In the discrete case, if c_0 and c_1 are identical and correspond to the Meixner measure, the matrix is also tridiagonal (see [5], [1]). When the matrix is tridiagonal, its characteristic polynomial satisfy a three term recurrence relation. To find the Markov-Bernstein constant is equivalent to find the smallest zero $\mu_{1,n}$ of the polynomial of degree n satisfying this three term recurrence relation, and $M_n = 1/\sqrt{\mu_{1,n}}$. The characteristic polynomial in the Jacobi case $c_0 = c_1 = c^{(\alpha,\beta)}$ provides a six-term recurrence relation (see [7]). Thus, to obtain a tridiagonal matrix is exceptional. There exists another case for which a tridiagonal matrix is obtained: it is the one of coherent pairs (c_0, c_1) .

The notion of coherent pairs of measures was introduced for first time by Iserles et al [8] in 1991. These measures are defined as follows

Definition 1.1 *Let c_0 and c_1 be two quasi definite linear functionals. Let $\{P_n\}_{n \geq 0}$ (resp. $\{T_n\}_{n \geq 0}$) be the sequence of monic orthogonal polynomials with respect to c_0 (resp. c_1). (c_0, c_1) is called coherent pair if and only if there exists a sequence $\{\sigma_n\}_{n \geq 1}$, $\sigma_n \in \mathbb{R}$, $\sigma_n \neq 0$, such that*

$$T_n(x) = \frac{P'_{n+1}(x)}{n+1} - \sigma_n \frac{P'_n(x)}{n}, \quad \forall n \geq 1. \quad (1)$$

All the kinds of coherent pairs (c_0, c_1) were described by Meijer [9] in 1997. They correspond to the relations (3.12) to (3.18) of his paper. One of both functionals is classical (Laguerre-Sonin or Jacobi). The seven cases contained in [9] are given below.

1. Laguerre case: $\Omega =]0, +\infty[$

- (a) c_0 corresponds to the measure $(x - \xi)x^{\alpha-1}e^{-x}dx$ with $\alpha > 0$ and $\xi < 0$.
 c_1 corresponds to the Laguerre-Sonin measure $x^\alpha e^{-x}dx$.
- (b) c_0 corresponds to the measure $e^{-x}dx + M\delta(0)$ with $M \geq 0$. δ is the Dirac measure.
 c_1 corresponds to the Laguerre measure $e^{-x}dx$.
- (c) c_0 corresponds to the Laguerre-Sonin measure $x^\alpha e^{-x}dx$ with $\alpha > -1$.
 c_1 corresponds to the measure $\frac{x^{\alpha+1}}{x-\xi}e^{-x}dx + M\delta(\xi)$ with $\xi \leq 0$ and $M \geq 0$.

2. Jacobi case: $\Omega =]-1, +1[$

- (a) c_0 corresponds to the measure $|x - \xi| (1-x)^{\alpha-1} (1+x)^{\beta-1} dx$ with $\alpha > 0$, $\beta > 0$ and $|\xi| > 1$.
 c_1 corresponds to the Jacobi measure $(1-x)^\alpha (1+x)^\beta dx$.
- (b) c_0 corresponds to the measure $(1+x)^{\beta-1} dx + M\delta(1)$ with $\beta > 0$ and $M \geq 0$.
 c_1 corresponds to the Jacobi measure $(1+x)^\beta dx$.
- (c) c_0 corresponds to the measure $(1-x)^{\alpha-1} dx + M\delta(-1)$ with $\alpha > 0$ and $M \geq 0$.
 c_1 corresponds to the Jacobi measure $(1-x)^\alpha dx$.
- (d) c_0 corresponds to the Jacobi measure $(1-x)^\alpha (1+x)^\beta dx$ with $\alpha > -1$ and $\beta > -1$.
 c_1 corresponds to the measure $\frac{1}{|x-\xi|} (1-x)^{\alpha+1} (1+x)^{\beta+1} dx + M\delta(\xi)$ with $|\xi| \geq 1$ and $M \geq 0$.

In every previous cases one measure depends on one parameter ξ or M , or both ξ and M . Our aim is to fix these parameters, and therefore to fix the different measures. From these measures we want to give, in an explicit form only depending on the fixed parameters, the two sequences of polynomials $\{P_n\}_{n \geq 0}$ and $\{T_n\}_{n \geq 0}$ as well as their L^2 norms, and the sequence $\{\sigma_n\}_{n \geq 1}$. After that part we want to obtain, also in an explicit form only depending on the fixed parameters, the three term recurrence relation linked with the study of the positivity of the bilinear functional $a_\lambda(p, q) = c_0(pq) + \lambda c_1(p'q')$ for $p, q \in \mathcal{P}$, where \mathcal{P} is the vector space of real polynomials in one variable.

We begin to give the general expression of such a three term recurrence relation.

Let us denote the square norm of P_n (resp. T_n) by $k_n^{(0)}$ (resp. $k_n^{(1)}$).

$$k_n^{(0)} = c_0(P_n^2) \text{ and } k_n^{(1)} = c_1(T_n^2), \quad \forall n \geq 0.$$

Like in several papers (see [3], [4], [6]) a particular basis of monic polynomials $\{R_i\}_{i \geq 0}^n$ of \mathcal{P}_n is defined from the following relations:

$$\begin{aligned} R'_0(x) &= 0, \\ R'_1(x) &= P'_1(x) = 1, \\ R'_i(x) &= iT_{i-1}(x) = P'_i(x) - \sigma_{i-1} \frac{i}{i-1} P'_{i-1}(x), \quad i = 2, \dots, n. \end{aligned}$$

From these relations the polynomials R_i are chosen as follows:

$$\begin{aligned} R_0(x) &= 1, \\ R_1(x) &= P_1(x), \\ R_i(x) &= P_i(x) - \sigma_{i-1} \frac{i}{i-1} P_{i-1}(x), \quad i = 2, \dots, n. \end{aligned}$$

Let p be a polynomial belonging to \mathcal{P}_n , written in this particular basis: $p(x) = \sum_{i=0}^n y_i R_i(x)$. $y_i \in \mathbb{R}$, $i = 0, \dots, n$. Let y be the vector of \mathbb{R}^{n+1} of components y_i , $i = 0, \dots, n$. Then

$$a_\lambda(p, p) = y^T K_{n,0}^{(0)} y + \lambda y^T K_{n,0}^{(1)} y$$

where $K_{n,0}^{(0)}$ is the $(n+1) \times (n+1)$ positive definite symmetric tridiagonal matrix

$$K_{n,0}^{(0)} = (c_0(R_j R_i))_{i,j=0}^n$$

and $K_{n,0}^{(1)}$ is the $(n+1) \times (n+1)$ positive semidefinite diagonal matrix

$$K_{n,0}^{(1)} = (c_1(R'_j R'_i))_{i,j=0}^n = (ij c_1(T_{j-1} T_{i-1}))_{i,j=0}^n.$$

By convention $T_{-1} = 0$.

Hence the content of both matrices $K_{n,0}^{(0)}$ and $K_{n,0}^{(1)}$.

$$\begin{aligned} K_{n,0}^{(1)}(i, i) &= i^2 c_1(T_{i-1}^2) = i^2 k_{i-1}^{(1)}, \quad i = 0, \dots, n. \\ K_{n,0}^{(0)}(i, i) &= k_i^{(0)}, \quad i = 0, 1, \\ &= k_i^{(0)} + \sigma_{i-1}^2 \left(\frac{i}{i-1}\right)^2 k_{i-1}^{(0)}, \quad i = 2, \dots, n. \\ K_{n,0}^{(0)}(i+1, i) = K_{n,0}^{(0)}(i, i+1) &= 0 \quad \text{if } i = 0, \\ &= -\sigma_i \left(\frac{i+1}{i}\right) k_i^{(0)}, \quad i = 1, \dots, n-1. \end{aligned}$$

We have the following obvious property

Property 1.2 $a_\lambda(p, p) > 0 \ \forall p \in \mathcal{P}_n$ if and only if $K_{n,0}^{(0)} + \lambda K_{n,0}^{(1)}$ is a positive definite symmetric matrix.

The first row and the first column of $K_{n,0}^{(0)} + \lambda K_{n,0}^{(1)}$ only contain one entry in row 0 and column 0, equal to $k_0^{(0)} > 0$. We may reduce the size of $K_{n,0}^{(0)} + \lambda K_{n,0}^{(1)}$ and only keep

$$\begin{aligned} K_{n,1}^{(0)} &= (c_0(R_j R_i))_{i,j=1}^n, \\ K_{n,1}^{(1)} &= (c_1(R'_j R'_i))_{i,j=1}^n. \end{aligned}$$

Then, Property 1.2 is equivalent to Property 1.3.

Property 1.3 $a_\lambda(p, p) > 0 \ \forall p \in \mathcal{P}_n$ if and only if $K_{n,1}^{(0)} + \lambda K_{n,1}^{(1)}$ is a positive definite symmetric matrix.

A transformation of the matrix $K_{n,1}^{(0)} + \lambda K_{n,1}^{(1)}$ uses the Cholesky decomposition of $K_{n,1}^{(0)}$.

$$K_{n,1}^{(0)} = G_{n,1}^{(0)} (G_{n,1}^{(0)})^T$$

where

$$\begin{aligned} G_{n,1}^{(0)}(i, i) &= \sqrt{k_i^{(0)}} \quad \text{if } i \geq 1, \\ G_{n,1}^{(0)}(i, i-1) &= -\sigma_{i-1} \frac{i}{i-1} \sqrt{k_{i-1}^{(0)}}, \quad i = 2, \dots, n. \end{aligned}$$

Then, we have

$$K_{n,1}^{(0)} + \lambda K_{n,1}^{(1)} = (K_{n,1}^{(1)})^{1/2} \left((K_{n,1}^{(1)})^{-1/2} G_{n,1}^{(0)} (G_{n,1}^{(0)})^T (K_{n,1}^{(1)})^{-1/2} + \lambda I \right) (K_{n,1}^{(1)})^{1/2}.$$

Let us denote $(K_{n,1}^{(1)})^{-1/2} G_{n,1}^{(0)} (G_{n,1}^{(0)})^T (K_{n,1}^{(1)})^{-1/2}$ by $\tilde{K}_{n,1}$ which is a $n \times n$ positive definite symmetric tridiagonal matrix. Its entries are

$$\begin{aligned} \tilde{K}_{n,1}(i, i) &= \frac{k_1^{(0)}}{k_0^{(1)}} \quad \text{if } i = 1, \\ &= \frac{k_i^{(0)}}{i^2 k_{i-1}^{(1)}} + \frac{\sigma_{i-1}^2}{(i-1)^2} \frac{k_{i-1}^{(0)}}{k_{i-1}^{(1)}}, \quad i = 2, \dots, n. \\ \tilde{K}_{n,1}(i+1, i) = \tilde{K}_{n,1}(i, i+1) &= -\frac{\sigma_i k_i^{(0)}}{i^2 \sqrt{k_{i-1}^{(1)} k_i^{(1)}}}, \quad i = 1, \dots, n-1. \end{aligned}$$

Property 1.3 implies that $\tilde{K}_{n,1} + \lambda I$ is positive definite. Thus $-\lambda$ has to be smaller than the smallest eigenvalue of $\tilde{K}_{n,1}$. The eigenvalues of $\tilde{K}_{n,1}$ also are the zeros of the characteristic polynomial $A_n(\lambda) = \det(\lambda I - \tilde{K}_{n,1})$. Of course the eigenvalues of $\tilde{K}_{n,1}$ are positive, so are the zeros of $A_n(\lambda)$. Since $\tilde{K}_{n,1}$ is a tridiagonal matrix, the polynomials $A_n(\lambda)$ satisfy a three term recurrence relation.

$$A_n(\lambda) = \left(\lambda - \frac{k_n^{(0)}}{n^2 k_{n-1}^{(1)}} - \frac{\sigma_{n-1}^2}{(n-1)^2} \frac{k_{n-1}^{(0)}}{k_{n-1}^{(1)}} \right) A_{n-1}(\lambda) - \frac{\sigma_{n-1}^2 (k_{n-1}^{(0)})^2}{(n-1)^4 k_{n-1}^{(1)} k_{n-2}^{(1)}} A_{n-2}(\lambda) \quad (2)$$

with $A_0(\lambda) = 1$ and $A_1(\lambda) = \lambda - \frac{k_1^{(0)}}{k_0^{(1)}}$.

This three term recurrence relation was also given by Pérez [16] in her thesis (Relation (5.4.13) page 157), but obtained by another way.

The smallest zero $\mu_{1,n}$ of $A_n(\lambda)$ provides the following Markov-Bernstein inequality

Theorem 1.4 *The following Markov-Bernstein inequality holds*

$$c_1((p')^2) \leq \frac{1}{\mu_{1,n}} c_0(p^2), \quad \forall p \in \mathcal{P}_n \quad (3)$$

where $\mu_{1,n}$ is the smallest zero $\mu_{1,n}$ of the polynomial $A_n(\lambda)$ satisfying the three term recurrence relation (2). Moreover the extremal polynomial \tilde{p} for which (3) becomes an equality, is given by

$$\tilde{p} = \sum_{j=1}^n \frac{w_j^{(1,n)}}{j \sqrt{k_{j-1}^{(1)}}} R_j \quad (4)$$

where $(w^{(1,n)})^T = (w_1^{(1,n)}, \dots, w_n^{(1,n)})$ is the eigenvector of $\tilde{K}_{n,1}$ associated to the eigenvalue $\mu_{1,n}$.

Proof.

Let p be a polynomial of \mathcal{P}_n written in the basis of the R_i 's. We look for the polynomials p such that $y^T(K_{n,0}^{(0)} - \lambda K_{n,0}^{(1)})y = 0$. Let \tilde{y} be the vector of \mathbb{R}^n such that $\tilde{y}^T = (y_1, \dots, y_n)$.

$$\begin{aligned} y^T(K_{n,0}^{(0)} - \lambda K_{n,0}^{(1)})y &= y^T \begin{pmatrix} k_0^{(0)} & 0 \\ 0 & K_{n,1}^{(0)} - \lambda K_{n,1}^{(1)} \end{pmatrix} y \\ &= k_0^{(0)} y_0^2 + \tilde{y}^T (K_{n,1}^{(0)} - \lambda K_{n,1}^{(1)}) \tilde{y} \\ &= k_0^{(0)} y_0^2 + \tilde{y}^T ((K_{n,1}^{(1)})^{1/2} (\tilde{K}_{n,1} - \lambda I) (K_{n,1}^{(1)})^{1/2}) \tilde{y} \end{aligned}$$

Therefore, if $w^{(1,n)} = (K_{n,1}^{(1)})^{1/2} \tilde{y}$ is the eigenvector of $\tilde{K}_{n,1}$ associated to the eigenvalue $\mu_{1,n}$ and if $y_0 = 0$, we have $y^T(K_{n,0}^{(0)} - \lambda K_{n,0}^{(1)})y = 0$. The entries y_j , for $j = 1, \dots, n$, are equal to $\frac{w_j^{(1,n)}}{j \sqrt{k_{j-1}^{(1)}}}$.

Hence (4) holds \square

In some cases an explicit upper bound of this constant can be obtained by using the Newton method on the polynomial $A_n(\lambda)$ (see [4], [6]). Indeed we have $0 < x_1 = -\frac{A_n(0)}{A_n'(0)} < \mu_{1,n}$. $A_n(0)$ is easily obtained from the Cholesky decomposition of $\tilde{K}_{n,1}$. $\tilde{K}_{n,1} = \tilde{G}_{n,1}(\tilde{G}_{n,1})^T$ with

$$\begin{aligned} \tilde{G}_{n,1}(i, i) &= \frac{1}{i} \sqrt{\frac{k_i^{(0)}}{k_{i-1}^{(1)}}}, \quad i = 1, \dots, n, \\ \tilde{G}_{n,1}(i, i-1) &= -\frac{\sigma_{i-1}}{i-1} \sqrt{\frac{k_{i-1}^{(0)}}{k_{i-1}^{(1)}}}, \quad i = 2, \dots, n. \end{aligned}$$

Since $A_n(\lambda) = \det(\lambda I - \tilde{K}_{n,1})$, we have

$$A_n(0) = (-1)^n (\det(\tilde{G}_{n,1}))^2 = (-1)^n \prod_{i=1}^n (\tilde{G}_{n,1}(i, i))^2 = (-1)^n \frac{1}{(n!)^2} \prod_{i=1}^n \frac{k_i^{(0)}}{k_{i-1}^{(1)}}. \quad (5)$$

The expression of $A_n'(0)$ is obtained from

$$A_n'(0) = A_{n-1}(0) - \left(\frac{k_n^{(0)}}{n^2 k_{n-1}^{(1)}} + \frac{\sigma_{n-1}^2}{(n-1)^2} \frac{k_{n-1}^{(0)}}{k_{n-1}^{(1)}} \right) A_{n-1}'(0) - \frac{\sigma_{n-1}^2 (k_{n-1}^{(0)})^2}{(n-1)^4 k_{n-1}^{(1)} k_{n-2}^{(1)}} A_{n-2}'(0) \quad (6)$$

with $A_0'(0) = 0$ and $A_1'(0) = 1$.

Moreover the qd algorithm, applied to the sequence $\{A_j(\lambda)\}_{j=1}^n$, also provides in some cases an

explicit lower bound of the Markov-Bernstein constant (see [6]). In this case the starting relations of this algorithm are

$$q_j^{(0)} = \frac{k_j^{(0)}}{j^2 k_{j-1}^{(1)}}, \quad j = 1, \dots, n, \quad (7)$$

$$e_{j-1}^{(0)} = \frac{\sigma_{j-1}^2}{(j-1)^2} \frac{k_{j-1}^{(0)}}{k_{j-1}^{(1)}}, \quad j = 2, \dots, n. \quad (8)$$

Then, next sequences $\{q_j^{(r)}\}$ and $\{e_j^{(r)}\}$ are computed as follows, for $r = 0, 1, \dots$

$$\begin{aligned} e_0^{(r)} &= e_n^{(r)} = 0, \\ q_{j+1}^{(r)} + e_{j+1}^{(r)} &= q_{j+1}^{(r+1)} + e_j^{(r+1)}, \quad j = 0, \dots, n-1, \\ q_{j+1}^{(r)} e_j^{(r)} &= q_j^{(r+1)} e_j^{(r+1)}, \quad j = 1, \dots, n-1 \end{aligned} \quad (9)$$

and the following inequalities hold (see [6] Corollary 2.4).

$$\mu_{1,n} < q_n^{(r)} < q_n^{(r-1)} < \dots < q_n^{(0)}. \quad (10)$$

Another way to obtain some informations about the zeros of $A_n(\lambda)$ is to use Blumenthal Theorem that we recall below in its simplified version (see Chihara [2] page 122, Theorem 4.1).

Theorem 1.5 *Let $A_n(\lambda) = (\lambda - B_n)A_{n-1}(\lambda) - C_n A_{n-2}(\lambda)$ be a three term recurrence relation in which*

$$\lim_{n \rightarrow \infty} B_n = \nu \quad \text{and} \quad \lim_{n \rightarrow \infty} C_n = \eta > 0.$$

Let us set $\sigma = \nu - 2\sqrt{\eta}$ and $\tau = \nu + 2\sqrt{\eta}$.

Then, the set X of all the zeros of all the polynomials $A_n(\lambda)$ is dense in $[\sigma, \tau]$.

Some isolated zeros can be lain on the outside of $[\sigma, \tau]$.

Remark 1.6 *If $\sigma = 0$ and $\eta > 0$, or if $\sigma = \eta = 0$, since all the zeros of $A_n(\lambda)$ are positive, we can conclude that $\lim_{n \rightarrow \infty} \mu_{1,n} = 0$.*

Before studying the different cases of coherent pairs, some properties of monic Laguerre-Sonin and Jacobi polynomials are recalled.

Monic Laguerre-Sonin polynomials $L_n^\alpha(x)$, $\alpha > -1$.

They satisfy the following three term recurrence relation (see Chihara [2] Relation (2.30) page 154):

$$L_{n+1}^\alpha(x) = (x - 2n - \alpha - 1)L_n^\alpha(x) - n(n + \alpha)L_{n-1}^\alpha(x), \quad \forall n \geq 1 \quad (11)$$

with $L_0^\alpha(x) = 1$ and $L_1^\alpha(x) = x - \alpha - 1$.

Their norm is

$$k_n^\alpha = \int_0^\infty (L_n^\alpha(x))^2 x^\alpha e^{-x} dx = n! \Gamma(\alpha + n + 1) \quad (12)$$

where Γ is the Gamma function.

The explicit formula of $L_n^\alpha(x)$ is

$$L_n^\alpha(x) = (-1)^n n! \sum_{m=0}^n \frac{(-1)^m}{m!} \binom{n+\alpha}{n-m} x^m \quad (13)$$

where $\binom{a}{n}$ denotes the binomial coefficient.

Moreover

$$\frac{d}{dx}L_n^\alpha(x) = nL_{n-1}^{\alpha+1}(x). \quad (14)$$

Monic Jacobi polynomials $P_n^{\alpha,\beta}(x)$, $\alpha > -1$, $\beta > -1$.

They satisfy the following three term recurrence relation (see Chihara [2] Relation (2.29) page 153):

$$\begin{aligned} P_{n+1}^{(\alpha,\beta)}(x) &= \left(x - \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)}\right) P_n^{(\alpha,\beta)}(x) \\ &\quad - \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)(2n + \alpha + \beta - 1)} P_{n-1}^{(\alpha,\beta)}(x) \end{aligned} \quad (15)$$

with $P_0^{(\alpha,\beta)}(x) = 1$ and $P_1^{(\alpha,\beta)}(x) = x - \frac{\beta - \alpha}{\alpha + \beta + 2}$.

Their norm is

$$\begin{aligned} k_n^{(\alpha,\beta)} &= \int_{-1}^1 (P_n^{(\alpha,\beta)}(x))^2 (1-x)^\alpha (1+x)^\beta dx \\ &= 2^{2n+\alpha+\beta+1} n! \frac{\Gamma(n + \alpha + 1) \Gamma(n + \beta + 1) \Gamma(n + \alpha + \beta + 1)}{(2n + \alpha + \beta + 1) (\Gamma(2n + \alpha + \beta + 1))^2}. \end{aligned} \quad (16)$$

Note that (16) could also be written for more convenience as

$$P_{n+1}^{(\alpha,\beta)}(x) = \left(x - \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)}\right) P_n^{(\alpha,\beta)}(x) - \frac{k_n^{(\alpha,\beta)}}{k_{n-1}^{(\alpha,\beta)}} P_{n-1}^{(\alpha,\beta)}(x). \quad (17)$$

We also have two other recurrence relations

$$P_n^{(\alpha,\beta)}(x) = P_n^{(\alpha+1,\beta)}(x) - \frac{2n(n + \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 1)} P_{n-1}^{(\alpha+1,\beta)}(x), \quad (18)$$

$$P_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta+1)}(x) + \frac{2n(n + \alpha)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 1)} P_{n-1}^{(\alpha,\beta+1)}(x), \quad (19)$$

Two explicit formulas will be used

$$P_n^{(\alpha,\beta)}(x) = 2^n \frac{\Gamma(\alpha + n + 1)}{\Gamma(\alpha + \beta + 2n + 1)} \sum_{m=0}^n \binom{n}{m} \frac{\Gamma(\alpha + \beta + n + m + 1)}{\Gamma(\alpha + m + 1)} \left(\frac{x-1}{2}\right)^m \quad (20)$$

$$= (-2)^n \frac{\Gamma(\beta + n + 1)}{\Gamma(\alpha + \beta + 2n + 1)} \sum_{m=0}^n \binom{n}{m} \frac{\Gamma(\alpha + \beta + n + m + 1)}{\Gamma(\beta + m + 1)} \left(-\frac{x+1}{2}\right)^m. \quad (21)$$

Moreover

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = n P_{n-1}^{(\alpha+1,\beta+1)}(x). \quad (22)$$

2 Laguerre case

2.1 Laguerre case (a)

The measure associated to c_0 is $(x - \xi)x^{\alpha-1}e^{-x}dx$ with $\alpha > 0$ and $\xi < 0$, and the one associated to c_1 is the Laguerre-Sonin measure $x^\alpha e^{-x}dx$. Therefore $T_n(x) = L_n^\alpha(x)$. The monic polynomials,

orthogonal with respect to c_1 , will be denoted by $P_n(x, \xi)$. Their expression, depending on the monic Laguerre polynomials $L_j^{\alpha-1}(x)$, has already been given in the thesis of Pérez [16] (Corollary 5.3.3 and Theorem 5.3.4). Nevertheless we give a direct proof of this expression. In this case we also obtain the square norm $k_n^{(0)}$ and σ_n .

Theorem 2.1 *The monic polynomials $P_n(x, \xi)$, orthogonal with respect to c_1 , satisfy the following relation:*

$$P_n(x, \xi) = N_n^{\alpha-1}(x, \xi) \frac{k_n^{\alpha-1}}{L_n^{\alpha-1}(\xi)} \quad (23)$$

where $N_n^{\alpha-1}(x, \xi)$ is the reproducing kernel associated to the measure $x^{\alpha-1}e^{-x}dx$.

$$N_n^{\alpha-1}(x, \xi) = \sum_{j=0}^n \frac{L_j^{\alpha-1}(x)L_j^{\alpha-1}(\xi)}{k_j^{\alpha-1}}.$$

The square norm $k_n^{(0)}$ of $P_n(x, \xi)$ and σ_n are given by

$$k_n^{(0)} = -n!\Gamma(\alpha+n) \frac{L_{n+1}^{\alpha-1}(\xi)}{L_n^{\alpha-1}(\xi)}, \quad \forall n \geq 0, \quad (24)$$

$$\sigma_n = n(n+\alpha) \frac{L_n^{\alpha-1}(\xi)}{L_{n+1}^{\alpha-1}(\xi)}, \quad \forall n \geq 1. \quad (25)$$

Proof.

Let us write $(x - \xi)P_n(x, \xi)$ in the basis of monic Laguerre polynomials $L_j^{\alpha-1}(x)$.

$$(x - \xi)P_n(x, \xi) = \sum_{j=0}^{n+1} \theta_{j,n+1} L_j^{\alpha-1}(x), \quad \theta_{n+1,n+1} = 1. \quad (26)$$

Let us denote by $c^{\alpha-1}$ the linear functional defined from the measure $x^{\alpha-1}e^{-x}$ on the support $\Omega =]0, +\infty[$. We have.

$$\begin{aligned} c^{\alpha-1}(L_i^{\alpha-1}(x)(x - \xi)P_n(x, \xi)) &= c_0(L_i^{\alpha-1}(x)P_n(x, \xi)) = 0 \text{ for } i = 0, \dots, n-1, \\ &= \sum_{j=0}^i \theta_{j,n+1} c^{\alpha-1}(L_i^{\alpha-1}(x)L_j^{\alpha-1}(x)) \\ &= \theta_{i,n+1} k_i^{\alpha-1}. \end{aligned}$$

Therefore $\theta_{i,n+1} = 0$ for $i = 0, \dots, n-1$, and

$$\theta_{n,n+1} = \frac{k_n^{(0)}}{k_n^{\alpha-1}}. \quad (27)$$

We put $x = \xi$ in (26). Then we obtain

$$\theta_{n,n+1} = -\frac{L_{n+1}^{\alpha-1}(\xi)}{L_n^{\alpha-1}(\xi)}. \quad (28)$$

The expression (24) of the square norm $k_n^{(0)}$ is obtained from (27) and (28). From (26) we have

$$P_n(x, \xi) = \left(L_{n+1}^{\alpha-1}(x) - \frac{L_{n+1}^{\alpha-1}(\xi)}{L_n^{\alpha-1}(\xi)} L_n^{\alpha-1}(x) \right) \frac{1}{x - \xi}. \quad (29)$$

(23) is obtained by using the definition of the reproducing kernel $N_n^{\alpha-1}(x, \xi)$ associated to the Christoffel-Darboux formula.

Now let us determine explicitly the sequence $\{\sigma_n\}_{n \geq 1}$ as a function depending on ξ . From (1), (14) and (23) we have

$$\begin{aligned}\sigma_n &= \frac{n}{n+1} \frac{P'_{n+1}(x, \xi) - (n+1)L_n^\alpha(x)}{P'_n(x, \xi)} \\ &= \frac{n}{n+1} \frac{1}{P'_n(x, \xi)} \sum_{j=1}^n j L_{j-1}^\alpha(x) \frac{L_j^{\alpha-1}(\xi)}{L_{n+1}^{\alpha-1}(\xi)} \frac{k_{n+1}^{\alpha-1}}{k_j^{\alpha-1}}\end{aligned}\quad (30)$$

To obtain σ_n it is sufficient to give the ratio of the coefficients of x^{n-1} in the numerator and the denominator of (30). Hence (25) holds. \square

Note that (25) is also given in [10].

At last the three term recurrence relation satisfied by the polynomials $A_n(\lambda)$ is given.

Theorem 2.2 *The following Markov-Bernstein inequality holds:*

$$c_1((p')^2) \leq \frac{1}{\mu_{1,n}} c_0(p^2), \quad \forall p \in \mathcal{P}_n$$

where $\mu_{1,n}$ is the smallest zero of the polynomials $A_n(\lambda)$ satisfying the following three term recurrence relation

$$A_n(\lambda) = (\lambda - 2 + \frac{\xi - \alpha}{n}) A_{n-1}(\lambda) - (1 + \frac{\alpha}{n-1}) A_{n-2}(\lambda), \quad \forall n \geq 2. \quad (31)$$

$A_0(\lambda) = 1$ and $A_1(\lambda) = \lambda - \alpha - 1 + \xi - \frac{\xi}{\xi - \alpha}$.

Proof.

We use the expressions of $k_n^{(0)}$ and σ_n in (2). We obtain

$$A_n(\lambda) = (\lambda + \frac{1}{n} \frac{L_{n+1}^{\alpha-1}(\xi)}{L_n^{\alpha-1}(\xi)} + (n + \alpha - 1) \frac{L_{n-1}^{\alpha-1}(\xi)}{L_n^{\alpha-1}(\xi)}) A_{n-1}(\lambda) - \frac{n + \alpha - 1}{n-1} A_{n-2}(\lambda), \quad \forall n \geq 2.$$

This relation is simplified by using the three term recurrence relation (11). Hence (31) holds. \square

Remark 2.3 *In the case of Markov-Bernstein inequalities associated to a bilinear functional $\tilde{a}_\lambda(p, q) = c_1(pq) + \lambda c_1(p'q')$ for $p, q \in \mathcal{P}$, the Markov-Bernstein constant $1/\sqrt{\mu_{1,n}}$ is obtained with the smallest zero $\mu_{1,n}$ of the polynomial $\tilde{A}_n(\lambda)$ satisfying the following three term recurrence relation (see Milovanović et al [13] page 576, Draux and Elhami [4] Theorem 6.3)*

$$\tilde{A}_n(\lambda) = (\lambda - 2 - \frac{\alpha}{n}) \tilde{A}_{n-1}(\lambda) - (1 + \frac{\alpha}{n-1}) \tilde{A}_{n-2}(\lambda), \quad \forall n \geq 2, \quad (32)$$

$\tilde{A}_0(\lambda) = 1$ and $\tilde{A}_1(\lambda) = \lambda - \alpha - 1$.

(31) tends to (32) when ξ tends to 0. ξ can be considered as a perturbation of the classical case $\tilde{a}_\lambda(p, q)$.

Moreover the following three term recurrence relation

$$p_n(\lambda) = (\lambda - 2) p_{n-1}(\lambda) - p_{n-2}(\lambda)$$

can be transformed as

$$p_n^*(y) = 2y p_{n-1}^*(y) - p_{n-2}^*(y)$$

by using $p_n^*(y) = p_n(\frac{\lambda-2}{2})$. We obtain the three term recurrence relation corresponding to Chebyshev polynomials. Therefore Relations (31) and (32) can also be considered as a perturbed Chebyshev three term recurrence relation by a ℓ^2 perturbation of the coefficients. Such perturbations of coefficients of a three term recurrence relation appear, for example, in relations studied in [1].

Remark 2.4 The fact that the polynomials $A_n(\lambda)$ satisfying Relation (32) are monic co-recursive Pollaczek polynomials, was already established by Pérez in her doctoral dissertation (see [16] page 28; see also Marcellán et al [12]). Monic co-recursive Pollaczek polynomials satisfy the three term recurrence relation of monic Pollaczek polynomials $P_n^\gamma(x; a, b, c)$ (see Chihara [2] page 185 Relation 5.9), but the polynomial of degree 1 is different.

$$\begin{aligned} P_n^\gamma(x; a, b, c) &= \left(x + \frac{b}{n + \gamma + a + c - 1}\right) P_{n-1}^\gamma(x; a, b, c) \\ &\quad - \frac{(n + c - 1)(n + 2\gamma + c - 2)}{(n + \gamma + a + c - 1)(n + \gamma + a + c - 2)} P_{n-2}^\gamma(x; a, b, c), \\ P_1^\gamma(x; a, b, c) &= x + \frac{b}{\gamma + a + c}. \end{aligned} \quad (33)$$

Indeed, if $A_n^*(y) = A_n(\frac{\lambda-2}{2})$, we have

$$\begin{aligned} A_n^*(y) &= \left(y + \frac{\xi - \alpha}{2n}\right) A_{n-1}^*(y) - \frac{1}{4} \left(1 + \frac{\alpha}{n-1}\right) A_{n-2}^*(y), \\ A_1^*(y) &= y + \frac{\xi - \alpha}{2} + \frac{\alpha}{2(\xi - \alpha)}. \end{aligned} \quad (34)$$

Relations (33) and (34) are identical if and only if

$$\gamma + a + c - 1 = 0, \quad b = \frac{\xi - \alpha}{2}, \quad \alpha = 2(\gamma + c - 1) \text{ and } (c - 1)(2\gamma + c - 2) = 0.$$

Therefore, either $c = 1$ or $c = 2(1 - \gamma)$.

If $c = 1$, then $\alpha = 2\gamma$, $a = -\gamma$ and $b = \frac{\xi - \alpha}{2}$.

If $c = 2(1 - \gamma)$, then $\alpha = c = 2(1 - \gamma)$, $a = \gamma - 1$ and $b = \frac{\xi - \alpha}{2}$.

Hence the corresponding Pollaczek polynomials are $P_n^\gamma(x; -\gamma, b, 1)$ or $P_n^\gamma(x; \gamma - 1, b, 2(1 - \gamma))$ with $b = \frac{\xi - \alpha}{2}$ (see also [1] for a use of these two families of polynomials). $A_1^*(y)$ is different from $P_1^\gamma(x; a, b, c) = x + \frac{b}{\gamma + a + c} = x + b = x + \frac{\xi - \alpha}{2}$.

Property 2.5

$$\lim_{n \rightarrow \infty} \mu_{1,n} = 0.$$

Proof.

We use Blumenthal Theorem 1.5. $\lim_{n \rightarrow \infty} B_n = 2$ and $\lim_{n \rightarrow \infty} C_n = 1$. Therefore $\sigma = 0$ and $\tau = 4$. Hence the property holds by using Remark 1.6. \square

2.2 Laguerre case (b)

The measure associated to c_0 is $e^{-x}dx + M\delta(0)$ with $M \geq 0$, and the one associated to c_1 is the Laguerre measure $e^{-x}dx$. Therefore $T_n(x)$ is the monic classical Laguerre polynomial $L_n(x)$. Let us denote by $P_n(x, M)$, $n \geq 0$, the monic polynomials orthogonal with respect to c_0 .

In order to obtain an explicit expression of $P_n(x, M)$, it will be written in the basis of monic Laguerre polynomial $L_j(x)$.

$$P_n(x, M) = \sum_{j=0}^n \theta_{j,n} L_j(x), \quad \theta_{n,n} = 1. \quad (35)$$

Property 2.6 *The explicit expressions of $P(x, M)$ in the basis of monic Laguerre polynomials $L_j(x)$, of its square norm $k_n^{(0)}$ and of σ_n are*

$$P_n(x, M) = L_n(x) + \sum_{j=0}^{n-1} (-1)^{n+1+j} \frac{n!M}{j!(nM+1)} L_j(x), \quad (36)$$

$$k_n^{(0)} = (n!)^2 \frac{(n+1)M+1}{nM+1}, \quad \forall n \geq 0, \quad (37)$$

$$\sigma_n = -\frac{n(1+nM)}{(n+1)M+1}, \quad \forall n \geq 1. \quad (38)$$

Proof.

To obtain the $\theta_{j,n}$'s we use the orthogonality relations of $P_n(x, M)$ for $i = 0, \dots, n-1$.

$$\begin{aligned} c_0(L_i(x)P_n(x, M)) &= \sum_{j=0}^n \theta_{j,n} (c_1(L_i(x)L_j(x)) + M(L_i(0)L_j(0))) = 0 \\ &= \theta_{i,n} k_i^{(1)} + M(-1)^i i! \sum_{j=0}^n \theta_{j,n} (-1)^j j! \end{aligned}$$

Thus

$$\theta_{i,n} = \frac{(-1)^{i+1}}{i!} M \sum_{j=0}^n \theta_{j,n} (-1)^j j! \quad \text{for } i = 0, \dots, n-1. \quad (39)$$

Let us set $K = \sum_{j=0}^n \theta_{j,n} (-1)^j j!$. Therefore

$$K = (-1)^n n! + \sum_{j=0}^{n-1} \theta_{j,n} (-1)^j j! = (-1)^n n! + \sum_{j=0}^{n-1} (-1)^{2j+1} M K$$

by using (39) in the definition of K . Hence $K = (-1)^n \frac{n!}{nM+1}$ and

$$\begin{aligned} \theta_{j,n} &= (-1)^{n+1+j} \frac{n!M}{j!(nM+1)}, \quad \text{for } j = 0, \dots, n-1, \\ \theta_{n,n} &= 1. \end{aligned}$$

Hence (36) holds.

Now we prove (37).

$$\begin{aligned} c_0((P_n(x, M))^2) &= c_1((P_n(x, M))^2) + M(P_n(0, M))^2 \\ &= k_n^{(1)} + \sum_{j=0}^{n-1} \left(\frac{n!M}{j!(nM+1)} \right)^2 k_j^{(1)} + M \left(L_n(0) + \sum_{j=0}^{n-1} (-1)^{n+1+j} \frac{n!M L_j(0)}{j!(nM+1)} \right)^2 \\ &= (n!)^2 \frac{(n+1)M+1}{nM+1}. \end{aligned}$$

At last σ_n is obtained from (1).

$$\sigma_n = \frac{n}{n+1} \frac{P'_{n+1}(x, M) - (n+1)L_n(x)}{P'_n(x, M)}.$$

It is sufficient to give the ratio of the coefficients of x^{n+1} of the numerator and of the denominator.

$$P'_{n+1}(x, M) = (n+1)L_n^1(x) + \sum_{j=1}^n (-1)^{n+2+j} \frac{(n+1)!M}{j!((n+1)M+1)} j L_{j-1}^1(x).$$

Its coefficient of x^{n+1} is $-n(n+1)\frac{n(n+2)M+n+1}{(n+1)M+1}$. The one of $(n+1)L_n(x)$ is $-n^2(n+1)$. Hence (38) holds. \square

If we rather use the following orthogonality relations $c_0(x^i P_n(x, M)) = 0$ for $i = 0, \dots, n-1$, two kinds of relations are obtained.

For $i = 0$

$$\theta_{0,n} + M \sum_{j=0}^n \theta_{j,n} L_j(0) = \theta_{0,n} + M \sum_{j=0}^n (-1)^j \theta_{j,n} j! = 0.$$

For $i = 1, \dots, n-1$,

$$\sum_{j=0}^i \theta_{j,n} c_1(x^i L_j(x)) = 0.$$

These last relations need to compute $c_1(x^i L_j(x))$ for $i \geq j$. It is much more complicated, but the expression of $c_1(x^i L_j(x))$ for $i \geq j$ is new. Moreover an original identity with a sum of products of binomial coefficients is deduced from this computation. It is the reason for which we give these results in the following Lemma and Corollary.

Lemma 2.7

$$\forall i \geq j, \quad c_1(x^i L_j(x)) = i! j! \binom{i}{j}. \quad (40)$$

Proof.

By using the formal expression of $L_n(x)$ given by (13) we have

$$c_1(x^i L_j(x)) = (-1)^j j! \sum_{m=0}^j \frac{(-1)^m}{m!} \binom{j}{m} c_1(x^{m+i}).$$

From the definition of the Gamma function we have

$$c_1(x^{m+i}) = \Gamma(m+i+1) = (m+i)!. \quad (41)$$

Therefore

$$c_1(x^i L_j(x)) = (-1)^j j! i! \sum_{m=0}^j (-1)^m \binom{j}{m} \binom{m+i}{i}. \quad (42)$$

For $i = j$, (40) corresponds to the square norm $k_j^{(1)} = (j!)^2$.

To prove (40) for $i > j$, we use the three term recurrence relation satisfied by the polynomials $L_j(x)$

$$L_{j+1}(x) = (x - 2j - 1)L_j(x) - j^2 L_{j-1}(x), \quad \forall j \geq 0. \quad (43)$$

This relation is multiplied by x^j and c_1 is applied to the result. We obtain

$$c_1(x^{j+1} L_j(x)) = (2j+1)!(j!)^2 + j^2 c_1(x^j L_{j-1}(x)), \quad \forall j \geq 0.$$

For $j = 1$ we have $c_1(x^{j+1} L_j(x)) = 4$. Therefore (40) holds for $i = j+1 = 2$. Then, by induction

$$\begin{aligned} c_1(x^{j+1} L_j(x)) &= (2j+1)(j!)^2 + j^2 j! (j-1)! \binom{j}{j-1} \\ &= j!(j+1)! \binom{j+1}{j}. \end{aligned}$$

Therefore (40) holds for $i = j+1, \forall j \geq 0$.

Now (40) is assumed to hold for $\ell \in \mathbb{N}$, ℓ fixed such that $i = j + \ell, \forall j \geq 0$.

$$c_1(x^{j+\ell} L_j(x)) = j!(j+\ell)! \binom{j+\ell}{j}, \quad \forall j \geq 0.$$

From (43) we have

$$\begin{aligned} c_1(x^{j+\ell+1}L_j(x)) &= c_1(x^{j+\ell}L_{j+1}(x)) + (2j+1)c_1(x^{j+\ell}L_j(x)) + j^2c_1(x^{j+\ell}L_{j-1}(x)) \\ &= (2j+\ell+1)j!(j+\ell)! \binom{j+\ell}{j} + j^2c_1(x^{j+\ell}L_{j-1}(x)). \end{aligned}$$

$c_1(x^{\ell+1}L_0(x)) = ((\ell+1)!)^2$, therefore (40) holds for $j = 0$ and $i = \ell + 1$. After that, by induction

$$\begin{aligned} c_1(x^{j+\ell+1}L_j(x)) &= (2j+\ell+1)j!(j+\ell)! \binom{j+\ell}{j} + j^2(j-1)!(j+\ell)! \binom{j+\ell}{j-1} \\ &= \frac{((j+\ell+1)!)^2}{(\ell+1)!} = j!(j+\ell+1)! \binom{j+\ell+1}{j}. \end{aligned}$$

Hence (40) holds. \square

From (42) the following corollary is deduced:

Corollary 2.8

$$(-1)^j \sum_{m=0}^j (-1)^m \binom{j}{m} \binom{m+i}{i} = \binom{i}{j}, \quad \forall i \geq j.$$

Finally, by using (37) and (38) in (2) we have the following theorem concerning the Markov-Bernstein inequalities.

Theorem 2.9 *The following Markov-Bernstein inequality holds:*

$$c_1((p')^2) \leq \frac{1}{\mu_{1,n}} c_0(p^2), \quad \forall p \in \mathcal{P}_n$$

where $\mu_{1,n}$ is the smallest zero of the polynomials $A_n(\lambda)$ satisfying the following three term recurrence relation

$$A_n(\lambda) = (\lambda - 2)A_{n-1}(\lambda) - A_{n-2}(\lambda), \quad \forall n \geq 2. \quad (44)$$

$$A_0(\lambda) = 1 \text{ and } A_1(\lambda) = \lambda - \frac{1+2M}{1+M}.$$

A lower bound of $\mu_{1,n}$ is obtained by using the Newton method.

Property 2.10 *The smallest zero $\mu_{1,n}$ of $A_n(\lambda)$ is such that*

$$\mu_{1,n} > \frac{1 + (n+1)M}{n(n+1)(3 + (n+2)M)}, \quad \forall n \geq 1. \quad (45)$$

Proof.

$A_n(0)$ is obtained from (5) and (37).

$$A_n(0) = (-1)^n \frac{1 + (n+1)M}{1 + M}.$$

$A'_n(0) = (-1)^{n+1} \frac{n(n+1)}{6(1+M)} (3 + (n+2)M)$ is obtained from (6) by using a proof by recurrence. Hence the result holds. \square

Then, the following inequality is deduced from the previous result

Corollary 2.11

$$c_1((p')^2) < \frac{n(n+1)(3 + (n+2)M)}{1 + (n+1)M} c_0(p^2), \quad \forall p \in \mathcal{P}_n.$$

A better lower bound of $\mu_{1,n}$ is provided by using the Laguerre method (see Draux [6]). Unfortunately this formal expression is very complicated. Nevertheless we give it.

$$0 < \tilde{x}_1 = -\frac{nA_n(0)}{A'_n(0) - \sqrt{H_n(0)}} < \mu_{1,n}$$

with $H_n(0) = (n-1)^2(A'_n(0))^2 - n(n-1)A_n(0)A''_n(0)$.
 $A''_n(0) = 2A'_n(0) - 2A''_{n-1}(0) - A''_{n-2}(0)$ and from this relation we obtain by recurrence

$$A''_n(0) = (-1)^n(n-1)_4 \frac{5 + (n+3)M}{60(1+M)}$$

where $(a)_n$ is the Pochhammer symbol: $(a)_n = a(a+1)\dots(a+n-1)$. By definition $(a)_0 = 1$. Then

$$\begin{aligned} \tilde{x}_1 &= \frac{1 + (n+1)M}{\frac{(n+1)(3+(n+2)M)}{6} - (-1)^{n+1}(n-1)\sqrt{K(n,M)}}, \\ K(n,M) &= \frac{(n+1)^2(3+(n+2)M)^2}{36} - \frac{1 + (n+1)M}{60}(n+1)(n+2)(5 + (n+3)M). \end{aligned}$$

At last we give the entries of the qd algorithm in order to obtain an upper bound of $\mu_{1,n}$. Relations (7) and (8) give the starting entries.

$$\begin{aligned} q_j^{(0)} &= \frac{1 + (j+1)M}{1 + jM}, \quad \text{for } j = 1, \dots, n, \\ e_j^{(0)} &= \frac{1 + jM}{1 + (j+1)M}, \quad \text{for } j = 1, \dots, n-1. \end{aligned}$$

$$\left\{ \begin{aligned} e_0^{(0)} = e_n^{(0)} &= 0, \\ q_j^{(1)} &= \frac{j+1}{j} \frac{1 + jM}{1 + (j+1)M} \frac{p_j^{(1)}}{p_{j-1}^{(1)}}, \quad j = 1, \dots, n-1, \\ e_j^{(1)} &= \frac{j}{j+1} \frac{1 + (j+2)M}{1 + (j+1)M} \frac{p_{j-1}^{(1)}}{p_j^{(1)}}, \quad j = 1, \dots, n-1, \\ q_n^{(1)} &= \frac{(1+nM)(1+(n+1)M)}{np_{n-1}^{(1)}}. \end{aligned} \right.$$

where $p_j^{(1)} = 1 + M(2+j) + \frac{M^2}{6}(2+j)(3+2j)$.

$$\left\{ \begin{aligned} e_0^{(1)} = e_n^{(1)} &= 0, \\ q_j^{(2)} &= \frac{(j+2)(2j+3)}{(j+1)(2j+1)} \frac{p_{j-1}^{(1)}}{p_j^{(1)}} \frac{p_j^{(2)}}{p_{j-1}^{(2)}}, \quad j = 1, \dots, n-1, \\ e_j^{(2)} &= \frac{j(2j+1)}{(j+1)(2j+3)} \frac{p_{j+1}^{(1)}}{p_j^{(1)}} \frac{p_{j-1}^{(2)}}{p_j^{(2)}}, \quad j = 1, \dots, n-2, \\ e_{n-1}^{(2)} &= \frac{(n-1)(2n-1)}{n(n+1)(2n+1)} \frac{(1+(n+1)M)^2}{p_{n-1}^{(1)}} \frac{p_{n-2}^{(2)}}{p_{n-1}^{(2)}}, \\ q_n^{(2)} &= \frac{30(1+(n+1)M)}{(n+1)(2n+1)} \frac{p_{n-1}^{(1)}}{p_{n-1}^{(2)}} \end{aligned} \right.$$

where $p_j^{(2)} = 5 + 5M(3+j) + M^2(3+j)(5+2j) + \frac{M^3}{6}(2+j)(3+j)(5+2j)$.

It is easy to see that all the entries $q_j^{(1)}$, $e_j^{(1)}$, $q_j^{(2)}$ and $e_j^{(2)}$ satisfy the set of relations (9).

As a consequence of (10) we have

Property 2.12 *The smallest zero $\mu_{1,n}$ of $A_n(\lambda)$ is such that*

$$\mu_{1,n} < q_n^{(2)}, \quad \forall n \geq 1. \quad (46)$$

Corollary 2.13 *The Markov-Bernstein constant M_n satisfies the following inequality*

$$\frac{1}{\sqrt{q_n^{(2)}}} \leq M_n \leq \sqrt{\frac{n(n+1)(3+(n+2)M)}{1+(n+1)M}}. \quad (47)$$

Corollary 2.14

$$\mu_{1,n} = O\left(\frac{1}{n^2}\right)$$

and the Markov-Bernstein constant has a behavior as a $O(n)$.

Below we give a table of some numerical results for the zero $\mu_{1,n}$ and the lower bounds x_1 and \tilde{x}_1 provided by the Newton method and the Laguerre method, as well as the upper bound $q_n^{(2)}$ provided by the qd algorithm, for different values of M .

M	n	x_1	\tilde{x}_1	$\mu_{1,n}$	$q_n^{(2)}$
1	20	0.002095238	0.019766736	0.020393972	0.029017408
	50	0.000370766	0.003519638	0.003649401	0.005391291
	100	0.000096181	0.000913322	0.000948578	0.001420806
	500	0.000003968	0.000037658	0.000039164	0.000059345
5	20	0.002233459	0.021242255	0.021922153	0.031139285
	50	0.000381719	0.003629865	0.003763805	0.005558176
	100	0.000097658	0.000927797	0.000963616	0.001443177
	500	0.000003980	0.000037778	0.000039289	0.000059534
10	20	0.002252829	0.021442114	0.022128520	0.031423693
	50	0.000383158	0.003644061	0.003778527	0.005579620
	100	0.000097848	0.000929632	0.000965523	0.001446012
	500	0.000003982	0.000037793	0.000039305	0.000059558
50	20	0.002268704	0.021604487	0.022296105	0.031654366
	50	0.000384322	0.003655484	0.003790372	0.005596869
	100	0.000098000	0.000931105	0.000967052	0.001448286
	500	0.000003983	0.000037805	0.000039317	0.000059577

2.3 Laguerre case (c)

The measure associated to c_0 is the Laguerre-Sonin measure $x^\alpha e^{-x} dx$ with $\alpha > -1$, and the one associated to c_1 is $\frac{x^{\alpha+1}}{x-\xi} e^{-x} dx + M\delta(\xi)$ with $\xi \leq 0$ and $M \geq 0$. Therefore the monic polynomials $P_n(x)$, orthogonal with respect to c_0 , are the monic Laguerre-Sonin polynomials $L_n^\alpha(x)$. Since we will also use the Laguerre-Sonin polynomials $L_n^{\alpha+1}(x)$, we prefer to denote by c^α the linear functional c_0 . Let us denote by $T_n(x, \xi, M)$, $n \geq 0$, the monic polynomials orthogonal with respect to c_1 . Moreover we will denote by \tilde{c}_1 the linear functional associated to the measure $\frac{x^{\alpha+1}}{x-\xi} e^{-x} dx$

and therefore, we have $c^{\alpha+1}(\cdot) = \tilde{c}_1((x - \xi)\cdot)$. Other notations will be used:

$$\begin{aligned} k_n^\alpha &= c^\alpha((L_n^\alpha(x))^2) = n!\Gamma(n + \alpha + 1), \\ \hat{k}_n &= c_1((T_n(x, \xi, M))^2). \end{aligned} \quad (48)$$

Relation (1) becomes

$$T_n(x, \xi, M) = L_n^{\alpha+1}(x) - \sigma_n L_{n-1}^{\alpha+1}(x). \quad (49)$$

Let us prove the following property giving a relation between $T_{n+1}(t, \xi, M)$ and the reproducing kernel $N_n^{\alpha+1}(x, t)$.

Property 2.15 *The monic polynomial $T_{n+1}(t, \xi, M)$, orthogonal with respect to c_1 , satisfies the following relation*

$$T_{n+1}(t, \xi, M) = -\frac{k_n^{\alpha+1}}{c_1(L_n^{\alpha+1}(x))} (1 - (t - \xi)c_1(N_n^{\alpha+1}(x, t))) \quad (50)$$

where the linear functional c_1 acts on the variable x and $N_n^{\alpha+1}(x, t)$ is the reproducing kernel

$$N_n^{\alpha+1}(x, t) = \sum_{j=0}^n \frac{L_j^{\alpha+1}(x)L_j^{\alpha+1}(t)}{k_j^{\alpha+1}}.$$

Proof.

$T_{n+1}(x, \xi, M)$ is written in the following basis $\{1, \{(x - \xi)L_j^{\alpha+1}(x)\}_{j=0}^n\}$. Thus

$$T_{n+1}(x, \xi, M) = T_{n+1}(\xi, \xi, M) + \sum_{j=0}^n \theta_{j,n} (x - \xi)L_j^{\alpha+1}(x), \quad \theta_{n,n} = 1. \quad (51)$$

(51) is multiplied by $L_i^{\alpha+1}(x)$ and c_1 is applied, for $i = 0, \dots, n$. We obtain

$$\begin{aligned} c_1(L_i^{\alpha+1}(x)T_{n+1}(x, \xi, M)) &= 0, \quad i = 0, \dots, n, \\ &= \tilde{c}_1(L_i^{\alpha+1}(x))T_{n+1}(\xi, \xi, M) + \sum_{j=0}^n \theta_{j,n} c^{\alpha+1}(L_i^{\alpha+1}(x)L_j^{\alpha+1}(x)) \\ &\quad + M L_i^{\alpha+1}(\xi) T_{n+1}(\xi, \xi, M) \\ &= T_{n+1}(\xi, \xi, M) (\tilde{c}_1(L_i^{\alpha+1}(x)) + M L_i^{\alpha+1}(\xi)) + \theta_{i,n} k_i^{\alpha+1} \\ &= T_{n+1}(\xi, \xi, M) c_1(L_i^{\alpha+1}(x)) + \theta_{i,n} k_i^{\alpha+1}. \end{aligned}$$

Hence

$$\theta_{i,n} = -\frac{T_{n+1}(\xi, \xi, M)}{k_i^{\alpha+1}} c_1(L_i^{\alpha+1}(x)), \quad i = 0, \dots, n, \quad (52)$$

and since $\theta_{n,n} = 1$ we have

$$T_{n+1}(\xi, \xi, M) = -\frac{k_n^{\alpha+1}}{c_1(L_n^{\alpha+1}(x))}. \quad (53)$$

Thus

$$\begin{aligned} T_{n+1}(t, \xi, M) &= -\frac{k_n^{\alpha+1}}{c_1(L_n^{\alpha+1}(x))} \left(1 - (t - \xi) \sum_{j=0}^n \frac{c_1(L_j^{\alpha+1}(x))L_j^{\alpha+1}(t)}{k_j^{\alpha+1}} \right) \\ &= -\frac{k_n^{\alpha+1}}{c_1(L_n^{\alpha+1}(x))} (1 - (t - \xi)c_1(N_n^{\alpha+1}(x, t))). \end{aligned}$$

□

Now relations giving \hat{k}_n and σ_n can be proved.

Property 2.16 *The square norm \hat{k}_n of the polynomials $T_n(x, \xi, M)$ and σ_n are given by*

$$\begin{aligned}\hat{k}_0 &= c_1(1), \\ \hat{k}_n &= -k_{n-1}^{\alpha+1} \frac{c_1(L_n^{\alpha+1}(x))}{c_1(L_{n-1}^{\alpha+1}(x))}, \quad \forall n \geq 1,\end{aligned}\tag{54}$$

$$\sigma_n = \frac{c_1(L_n^{\alpha+1}(x))}{c_1(L_{n-1}^{\alpha+1}(x))}, \quad \forall n \geq 1.\tag{55}$$

Proof.

$$\begin{aligned}\hat{k}_n &= c_1(L_n^{\alpha+1}(x)T_n(x, \xi, M)) \\ &= c_1(L_n^{\alpha+1}(x)T_n(\xi, \xi, M)) + c_1(L_n^{\alpha+1}(x) \sum_{j=0}^{n-1} \theta_{j,n}(x - \xi)L_j^{\alpha+1}(x)) \\ &= c_1(L_n^{\alpha+1}(x)T_n(\xi, \xi, M)).\end{aligned}$$

By using (53), (54) holds. Now

$$\begin{aligned}\hat{k}_n &= c_1((x - \xi)L_{n-1}^{\alpha+1}(x)T_n(x, \xi, M)) \\ &= c^{\alpha+1}(L_{n-1}^{\alpha+1}(x)T_n(x, \xi, M)) \\ &= -\sigma_n k_{n-1}^{\alpha+1} \text{ by using (49).} \\ \sigma_n &= -\frac{\hat{k}_n}{k_{n-1}^{\alpha+1}} = \frac{c_1(L_n^{\alpha+1}(x))}{c_1(L_{n-1}^{\alpha+1}(x))} \text{ by using (54).}\end{aligned}$$

□

At last we have the following theorem concerning the Markov-Bernstein inequalities.

Theorem 2.17 *The following Markov-Bernstein inequality holds:*

$$c_1((p')^2) \leq \frac{1}{\mu_{1,n}} c_0(p^2), \quad \forall p \in \mathcal{P}_n$$

where $\mu_{1,n}$ is the smallest zero of the polynomials $A_n(\lambda)$ satisfying the following three term recurrence relation

$$A_n(\lambda) = (\lambda + B_n)A_{n-1}(\lambda) - C_n A_{n-2}(\lambda), \quad \forall n \geq 2.\tag{56}$$

where

$$B_n = \frac{(n-1)(n+\alpha)}{n} \frac{c_1(L_{n-2}^{\alpha+1}(x))}{c_1(L_{n-1}^{\alpha+1}(x))} + \frac{1}{n-1} \frac{c_1(L_{n-1}^{\alpha+1}(x))}{c_1(L_{n-2}^{\alpha+1}(x))},\tag{57}$$

$$C_n = \frac{(n+\alpha-1)(n-2)}{(n-1)^2} \frac{c_1(L_{n-1}^{\alpha+1}(x))c_1(L_{n-3}^{\alpha+1}(x))}{(c_1(L_{n-2}^{\alpha+1}(x)))^2}.\tag{58}$$

$A_0(\lambda) = 1$ and

$$A_1(\lambda) = \lambda - \frac{k_1^\alpha}{c_1(1)} = \lambda - \frac{\Gamma(\alpha+2)}{c_1(1)}.\tag{59}$$

Proof.

By using (54), (55) and (48) we have

$$\begin{aligned}
B_n &= -\frac{k_n^\alpha}{n^2 \hat{k}_{n-1}} - \frac{\sigma_{n-1}^2}{(n-1)^2} \frac{k_{n-1}^\alpha}{\hat{k}_{n-1}} \\
&= \frac{(n-1)(n+\alpha)}{n} \frac{c_1(L_{n-2}^{\alpha+1}(x))}{c_1(L_{n-1}^{\alpha+1}(x))} + \frac{1}{n-1} \frac{c_1(L_{n-1}^{\alpha+1}(x))}{c_1(L_{n-2}^{\alpha+1}(x))}, \\
C_n &= \frac{\sigma_{n-1}^2 (k_{n-1}^\alpha)^2}{(n-1)^4 \hat{k}_{n-1} \hat{k}_{n-2}} \\
&= \frac{(n+\alpha-1)(n-2)}{(n-1)^2} \frac{c_1(L_{n-1}^{\alpha+1}(x)) c_1(L_{n-3}^{\alpha+1}(x))}{(c_1(L_{n-2}^{\alpha+1}(x)))^2}.
\end{aligned}$$

□

Note that $c_1(L_i^{\alpha+1}(x)) = \tilde{c}_1(L_i^{\alpha+1}(x)) + M L_i^{\alpha+1}(\xi)$. Therefore to compute $c_1(L_i^{\alpha+1}(x))$ is reduced to compute $\tilde{c}_1(L_i^{\alpha+1}(x))$.

Remark 2.18 If $\xi = 0$ and $M = 0$, (56) is

$$A_n(\lambda) = (\lambda - 2 - \frac{\alpha}{n}) A_{n-1}(\lambda) - (1 + \frac{\alpha}{n-1}) A_{n-2}(\lambda), \quad \forall n \geq 2, \quad (60)$$

Indeed $c_1(L_i^{\alpha+1}(x)) = \tilde{c}_1(L_i^{\alpha+1}(x))$ and by using the relation $L_i^\alpha(x) = L_i^{\alpha+1}(x) + i L_{i-1}^{\alpha+1}(x)$ we have

$$\tilde{c}_1(L_i^{\alpha+1}(x)) = -i \tilde{c}_1(L_{i-1}^{\alpha+1}(x)) = (-1)^i i! \tilde{c}_1(L_0^{\alpha+1}(x)) = (-1)^i i! \Gamma(\alpha+1). \quad (61)$$

This last relation, used in (57) and (58), provides the values of B_n and C_n of (60). (60) is the relation obtained in the study of Markov-Bernstein inequalities (see Remark 2.3) for the Laguerre-Sonin measure.

Remark 2.19 The Laguerre function of the second kind $Q_n^{\alpha+1}(\xi)$ can be used to replace the terms $c_1(L_n^{\alpha+1}(x))$ in B_n and C_n given in Theorem 2.17. $Q_n^\alpha(\xi)$ is defined as follows (see [14] Relation (5) page 287) for $\xi < 0$:

$$\begin{aligned}
Q_n^\alpha(\xi) &= \frac{1}{e^{-\xi} \xi^\alpha} \int_0^\infty \frac{\tilde{L}_n^\alpha(x)}{x - \xi} e^{-x} x^\alpha dx \\
&= e^{-i\pi\alpha} \Gamma(n + \alpha + 1) e^\xi G(n + \alpha + 1, \alpha + 1; -\xi)
\end{aligned}$$

where $\tilde{L}_n^\alpha(x)$ is the non-monic Laguerre polynomial ($\tilde{L}_n^\alpha(x) = \frac{(-1)^n}{n!} L_n^\alpha(x)$) and G is the confluent hypergeometric function of the second kind given by (see [14] Relation (16) page 272)

$$G(\alpha, \gamma; z) = \frac{\Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)} F(\alpha, \gamma; z) + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} z^{1-\gamma} F(\alpha+1-\gamma, 2-\gamma; z)$$

and $F(\alpha, \gamma; z) = \sum_{\nu=0}^\infty \frac{(\alpha)_\nu}{(\gamma)_\nu \nu!} z^\nu$.

Unfortunately with this function $Q_n^{\alpha+1}(\xi)$ the coefficients B_n and C_n do not become easier to use.

Remark 2.20 Now we will use the asymptotic behavior of $L_n^{\alpha+1}(\xi)$ and of $Q_n^{\alpha+1}(\xi)$ to compute σ and τ given in Blumenthal Theorem 1.5. The asymptotic behavior of $L_n^{\alpha+1}(\xi)$ was given by Perron (see Szegő [17] Theorem 8.22.3 page 199). For $\xi < 0$ we have

$$L_n^{\alpha+1}(\xi) = \frac{e^{\xi/2}}{2\sqrt{\pi}} \frac{e^{2\sqrt{-n\xi}} n^{\alpha/2+1/4}}{(-\xi)^{\alpha/2+3/4}} (1 + O(\frac{1}{\sqrt{n}})). \quad (62)$$

The asymptotic behavior of $\tilde{c}_1(L_n^{\alpha+1}(x))$ can be found in the paper of Meijer et al ([11] Lemma 4.3). For $\xi < 0$ we have

$$\tilde{c}_1(L_n^{\alpha+1}(x)) \sim e^{-\xi/2} \sqrt{\pi} e^{-2\sqrt{-n\xi}} n^{\alpha/2+1/4} (-\xi)^{\alpha/2+1/4}. \quad (63)$$

If $M > 0$, then $c_1(L_n^{\alpha+1}(x)) = \tilde{c}_1(L_n^{\alpha+1}(x)) + ML_n^{\alpha+1}(\xi) \sim ML_n^{\alpha+1}(\xi)$ and

$$\frac{c_1(L_{n-1}^{\alpha+1}(\xi))}{c_1(L_{n-2}^{\alpha+1}(\xi))} \sim \frac{L_{n-1}^{\alpha+1}(\xi)}{L_{n-2}^{\alpha+1}(\xi)} \sim \left(\frac{n-1}{n-2}\right)^{\alpha/2+1/4} e^{\sqrt{-\xi/(n-1)}} \rightarrow 1 \text{ when } n \rightarrow \infty.$$

Therefore $\lim_{n \rightarrow \infty} B_n = \infty$ and $\lim_{n \rightarrow \infty} C_n = 1$. Hence $\sigma = \tau = \infty$.

If $M = 0$, then $c_1(L_n^{\alpha+1}(x)) = \tilde{c}_1(L_n^{\alpha+1}(x))$ and

$$\frac{c_1(L_{n-1}^{\alpha+1}(\xi))}{c_1(L_{n-2}^{\alpha+1}(\xi))} \sim \frac{L_{n-1}^{\alpha+1}(\xi)}{L_{n-2}^{\alpha+1}(\xi)} \sim \left(\frac{n-1}{n-2}\right)^{\alpha/2+1/4} e^{-\sqrt{-\xi/(n-1)}} \rightarrow 1 \text{ when } n \rightarrow \infty.$$

Therefore we have the same result.

Property 2.21 If $\xi = 0$, then

$$\lim_{n \rightarrow \infty} \mu_{1,n} = 0.$$

Proof.

In this case $\tilde{c}_1 = c_0 = c^\alpha$. By using the relation $L_j^\alpha(x) = L_j^{\alpha+1}(x) + jL_{j-1}^{\alpha+1}(x)$ for $j = n, n-1, \dots, 1$ we obtain $c^\alpha(L_n^{\alpha+1}(x)) = (-1)^n n! k_0^\alpha = (-1)^n n! \Gamma(\alpha+1)$.

For $M > 0$ we have $c_1(L_n^{\alpha+1}(x)) = (-1)^n (n! \Gamma(\alpha+1) + M(\alpha+1)_n)$. Therefore

$$\begin{aligned} c_1(L_n^{\alpha+1}(x)) &\sim (-1)^n n! \Gamma(\alpha+1) && \text{if } -1 < \alpha < 0, \\ &= (-1)^n n! (M+1) && \text{if } \alpha = 0, \\ &\sim (-1)^n M(\alpha+1)_n && \text{if } \alpha > 0. \end{aligned}$$

In the three cases we have $\lim_{n \rightarrow \infty} -B_n = 2$ and $\lim_{n \rightarrow \infty} C_n = 1$. We use Blumenthal Theorem 1.5: we have $\sigma = 0$ and $\tau = 4$. Hence the property holds by using Remark 1.6.

If $M = 0$, the property also holds (see Remark 2.18). In this case the values of $\sigma = 0$ and $\tau = 4$ are obvious. \square

3 Jacobi case

3.1 Jacobi case (a)

The measure associated to c_0 is $|x - \xi| (1-x)^{\alpha-1} (1+x)^{\beta-1} dx$ with $\alpha > 0, \beta > 0$ and $|\xi| > 1$, and the one associated to c_1 is the Jacobi measure $(1-x)^\alpha (1+x)^\beta dx$. Therefore $T_n(x)$ is the monic Jacobi polynomial $P_n^{(\alpha, \beta)}(x)$. For more convenience the measure associated to c_0 will be written as $\varepsilon(x - \xi)(1-x)^{\alpha-1} (1+x)^{\beta-1} dx$ with $\varepsilon = 1$ if $\xi < -1$ and $\varepsilon = -1$ if $\xi > 1$.

The monic polynomials, orthogonal with respect to c_0 , will be denoted by $P_n(x, \xi)$. Their expression, depending on the monic Jacobi polynomials $P_n^{(\alpha-1, \beta-1)}(x)$, has already also been given in the thesis of Pérez [16] (Corollary 5.3.3 and Theorem 5.3.4). Also here we give a direct proof of this expression because we will also obtain the square norm $k_n^{(0)}$ and σ_n .

Theorem 3.1 The monic polynomials $P_n(x, \xi)$, orthogonal with respect to c_0 , satisfy the following relation:

$$P_n(x, \xi) = N_n^{(\alpha-1, \beta-1)}(x, \xi) \frac{k_n^{(\alpha-1, \beta-1)}}{P_n^{(\alpha-1, \beta-1)}(\xi)} \quad (64)$$

where $N_n^{(\alpha-1, \beta-1)}(x, \xi)$ is the reproducing kernel associated to the measure $(1-x)^{\alpha-1}(1+x)^{\beta-1}dx$.

$$N_n^{(\alpha-1, \beta-1)}(x, \xi) = \sum_{j=0}^n \frac{P_j^{(\alpha-1, \beta-1)}(x) P_j^{(\alpha-1, \beta-1)}(\xi)}{k_j^{(\alpha-1, \beta-1)}}.$$

The square norm $k_n^{(0)}$ of $P_n(x, \xi)$ and σ_n are given by

$$k_n^{(0)} = -\varepsilon k_n^{(\alpha-1, \beta-1)} \frac{P_{n+1}^{(\alpha-1, \beta-1)}(\xi)}{P_n^{(\alpha-1, \beta-1)}(\xi)}, \quad \forall n \geq 0, \text{ with } \varepsilon = 1 \text{ if } \xi < -1 \text{ and } \varepsilon = -1 \text{ if } \xi > 1 \quad (65)$$

$$\sigma_n = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta-1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)} \frac{P_n^{(\alpha-1, \beta-1)}(\xi)}{P_{n+1}^{(\alpha-1, \beta-1)}(\xi)}, \quad \forall n \geq 1 \quad (66)$$

where $k_n^{(\alpha-1, \beta-1)}$ is the square norm of the polynomials $P_n^{(\alpha-1, \beta-1)}(x)$ with respect to $(1-x)^{\alpha-1}(1+x)^{\beta-1}dx$.

Proof.

$(x - \xi)P_n(x, \xi)$ is written in the basis of monic Jacobi polynomials $P_j^{(\alpha-1, \beta-1)}(x)$.

$$(x - \xi)P_n(x, \xi) = \sum_{j=0}^{n+1} \theta_{j, n+1} P_j^{(\alpha-1, \beta-1)}(x), \quad \theta_{n+1, n+1} = 1. \quad (67)$$

Let us denote by $c^{(\alpha-1, \beta-1)}$ the linear functional defined from the Jacobi measure $(1-x)^{\alpha-1}(1+x)^{\beta-1}dx$ on the support $\Omega =]-1, 1[$. We have

$$\begin{aligned} c^{(\alpha-1, \beta-1)}(\varepsilon P_i^{(\alpha-1, \beta-1)}(x)(x - \xi)P_n(x, \xi)) &= c_0(P_i^{(\alpha-1, \beta-1)}(x)P_n(x, \xi)) = 0, \quad i = 0, \dots, n-1 \\ &= \varepsilon \sum_{j=0}^i \theta_{j, n+1} c^{(\alpha-1, \beta-1)}(P_i^{(\alpha-1, \beta-1)}(x)P_j^{(\alpha-1, \beta-1)}(x)) \\ &= \theta_{i, n+1} k_i^{(\alpha-1, \beta-1)}. \end{aligned}$$

Therefore $\theta_{i, n+1} = 0$ for $i = 0, \dots, n-1$ and

$$\theta_{n, n+1} = \frac{k_n^{(0)}}{\varepsilon k_n^{(\alpha-1, \beta-1)}}. \quad (68)$$

We put $x = \xi$ in (67). Then we obtain

$$\theta_{n, n+1} = -\frac{P_{n+1}^{(\alpha-1, \beta-1)}(\xi)}{P_n^{(\alpha-1, \beta-1)}(\xi)}. \quad (69)$$

The expression (65) of the square norm $k_n^{(0)}$ is deduced from (68) and (69). From (67) we have

$$P_n(x, \xi) = \left(P_{n+1}^{(\alpha-1, \beta-1)}(x) - \frac{P_{n+1}^{(\alpha-1, \beta-1)}(\xi)}{P_n^{(\alpha-1, \beta-1)}(\xi)} P_n^{(\alpha-1, \beta-1)}(x) \right) \frac{1}{x - \xi}. \quad (70)$$

Again, by using the definition of the reproducing kernel $N_n^{(\alpha-1, \beta-1)}(x, \xi)$ associated to the Christoffel-Darboux formula, (64) is obtained.

Note that (70) is also given in Lemma 3.2 of the paper of Pan [15].

Now let us determine explicitly the sequence $\{\sigma_n\}_{n \geq 1}$ as a function depending on ξ . From (1), (22) and (64) we have

$$\begin{aligned}\sigma_n &= \frac{n}{n+1} \frac{P'_{n+1}(x, \xi) - (n+1)P_n^{(\alpha, \beta)}(x)}{P'_n(x, \xi)} \\ &= \frac{n}{(n+1)P'_n(x, \xi)} \sum_{j=1}^n j P_{j-1}^{(\alpha, \beta)}(x) \frac{P_j^{(\alpha-1, \beta-1)}(\xi)}{P_{n+1}^{(\alpha-1, \beta-1)}(\xi)} \frac{k_{n+1}^{(\alpha-1, \beta-1)}}{k_j^{(\alpha-1, \beta-1)}}.\end{aligned}\quad (71)$$

(66) is deduced by using the ratio of the coefficients of x^{n+1} in the numerator and the denominator of (71).

$$\sigma_n = \frac{n}{(n+1)} \frac{P_n^{(\alpha-1, \beta-1)}(\xi)}{P_{n+1}^{(\alpha-1, \beta-1)}(\xi)} \frac{k_{n+1}^{(\alpha-1, \beta-1)}}{k_n^{(\alpha-1, \beta-1)}}. \quad (72)$$

□

Note that (66) is also given in [10].

Moreover note that (72) and (65) give a relation between $k_n^{(0)}$ and σ_n .

$$\sigma_n = -\varepsilon \frac{n}{(n+1)} \frac{k_{n+1}^{(\alpha-1, \beta-1)}}{k_n^{(0)}}. \quad (73)$$

At last we have the following theorem concerning the Markov-Bernstein inequalities.

Theorem 3.2 *The following Markov-Bernstein inequality holds:*

$$c_1((p')^2) \leq \frac{1}{\mu_{1,n}} c_0(p^2), \quad \forall p \in \mathcal{P}_n$$

where $\mu_{1,n}$ is the smallest zero of the polynomials $A_n(\lambda)$ satisfying the following three term recurrence relation

$$A_n(\lambda) = (\lambda + B_n)A_{n-1}(\lambda) - C_n A_{n-2}(\lambda), \quad \forall n \geq 2. \quad (74)$$

where

$$B_n = \frac{\varepsilon}{n(n+\alpha+\beta-1)} \left(\xi - \frac{(\beta-\alpha)(\alpha+\beta-2)}{(2n+\alpha+\beta-2)(2n+\alpha+\beta)} \right), \quad (75)$$

$$C_n = \frac{4(n+\alpha-1)(n+\beta-1)}{(n-1)(n+\alpha+\beta-1)(2n+\alpha+\beta-1)(2n+\alpha+\beta-2)^2(2n+\alpha+\beta-3)}. \quad (76)$$

$A_0(\lambda) = 1$ and

$$\begin{aligned}A_1(\lambda) &= \lambda + \frac{\varepsilon}{\alpha+\beta} \frac{P_2^{(\alpha-1, \beta-1)}(\xi)}{P_1^{(\alpha-1, \beta-1)}(\xi)} \\ &= \lambda + \frac{\varepsilon}{\alpha+\beta} \left(\xi - \frac{(\beta-\alpha)(\alpha+\beta-2)}{(2+\alpha+\beta)(\alpha+\beta)} - \frac{4\alpha\beta}{(\alpha+\beta)_2(\xi(\alpha+\beta) - \beta + \alpha)} \right) \\ &= \lambda + \varepsilon \frac{(1+\alpha+\beta)_2 \xi^2 + 2(\alpha-\beta)(1+\alpha+\beta)\xi + (\alpha-\beta)^2 - (2+\alpha+\beta)}{(1+\alpha+\beta)_2(\xi(\alpha+\beta) - \beta + \alpha)}\end{aligned}$$

Proof.

By using (73),

$$C_n = \frac{\sigma_{n-1}^2 (k_{n-1}^{(0)})^2}{(n-1)^4 k_{n-1}^{(\alpha, \beta)} k_{n-2}^{(\alpha, \beta)}} = \frac{(k_n^{(\alpha-1, \beta-1)})^2}{n^2 (n-1)^2 k_{n-1}^{(\alpha, \beta)} k_{n-2}^{(\alpha, \beta)}}.$$

Hence (76) holds.

$$\begin{aligned}
B_n &= -\frac{k_n^{(0)}}{n^2 k_{n-1}^{(\alpha, \beta)}} - \frac{\sigma_{n-1}^2}{(n-1)^2} \frac{k_{n-1}^{(0)}}{k_{n-1}^{(\alpha, \beta)}} \\
&= \frac{\varepsilon}{n^2} \frac{k_n^{(\alpha-1, \beta-1)}}{k_{n-1}^{(\alpha, \beta)}} \left(\frac{P_{n+1}^{(\alpha-1, \beta-1)}(\xi)}{P_n^{(\alpha-1, \beta-1)}(\xi)} + \frac{P_{n-1}^{(\alpha-1, \beta-1)}(\xi)}{P_n^{(\alpha-1, \beta-1)}(\xi)} \frac{k_n^{(\alpha-1, \beta-1)}}{k_{n-1}^{(\alpha-1, \beta-1)}} \right)
\end{aligned}$$

by using (72) and (65).

Now, by using the three term recurrence relation (17) the quantity inside the brackets is equal to

$$\xi - \frac{(\beta - \alpha)(\alpha + \beta - 2)}{(2n + \alpha + \beta - 2)(2n + \alpha + \beta)}.$$

Hence (75) holds. At last

$$\begin{aligned}
A_1(\lambda) &= \lambda - \frac{k_1^{(0)}}{k_0^{(\alpha, \beta)}} \\
&= \lambda + \frac{\varepsilon}{\alpha + \beta} \frac{P_2^{(\alpha-1, \beta-1)}(\xi)}{P_1^{(\alpha-1, \beta-1)}(\xi)} \\
&= \lambda + \frac{\varepsilon}{\alpha + \beta} \left(\xi - \frac{(\beta - \alpha)(\alpha + \beta - 2)}{(2 + \alpha + \beta)(\alpha + \beta)} - \frac{4\alpha\beta}{(1 + \alpha + \beta)(\alpha + \beta)(\xi(\alpha + \beta) - \beta + \alpha)} \right)
\end{aligned}$$

by using the three term recurrence relation (15) satisfied by the polynomials $P_2^{(\alpha-1, \beta-1)}(\xi)$, $P_1^{(\alpha-1, \beta-1)}(\xi)$ and $P_0^{(\alpha-1, \beta-1)}(\xi)$. \square

Property 3.3

$$\lim_{n \rightarrow \infty} \mu_{1,n} = 0.$$

Proof.

We use Blumenthal Theorem 1.5. $\lim_{n \rightarrow \infty} B_n = 0$ and $\lim_{n \rightarrow \infty} C_n = 0$. Therefore $\sigma = 0$ and $\tau = 0$. Hence the property holds by using Remark 1.6. \square

3.2 Jacobi case (b) and (c)

In the case (b) the measure associated to c_0 is $(1+x)^{\beta-1}dx + M\delta(1)$ with $\beta > 0$ and $M \geq 0$, and the one associated to c_1 is the Jacobi measure $(1+x)^\beta dx$. In the case (c) the measure associated to c_0 is $(1-x)^{\alpha-1}dx + M\delta(-1)$ with $\alpha > 0$ and $M \geq 0$, and the one associated to c_1 is the Jacobi measure $(1-x)^\alpha dx$. For more convenience and to give unique proofs for both cases (b) and (c) we will adopt a specific notation. The measure associated to c_0 will be $(1+\varepsilon x)^{\gamma-1}dx + M\delta(\varepsilon)$ with $\gamma > 0$ and $M \geq 0$, and the one associated to c_1 will be $(1+\varepsilon x)^\gamma dx$. If $\varepsilon = 1$ then $\gamma = \beta$, and if $\varepsilon = -1$ then $\gamma = \alpha$. The monic polynomials, orthogonal with respect to c_0 , will be denoted by $P_n(x, M)$. The monic polynomials $T_n(x)$, orthogonal with respect to c_1 , will be denoted by $P_n^{(\gamma)}$. If $\varepsilon = 1$, this polynomial is the monic Jacobi polynomial $P_n^{(0, \beta)}(x)$, and if $\varepsilon = -1$, this polynomial is the monic Jacobi polynomial $P_n^{(\alpha, 0)}(x)$. Thanks to this specific notation we have unique expressions for the three term recurrence relation satisfied by the polynomials $P_n^{(\gamma)}$, for the

square norm $k_n^{(\gamma)}$, and for the explicit formula of $P_n^{(\gamma)}$ corresponding to (20) and (21).

$$P_{n+1}^{(\gamma)}(x) = \left(x - \frac{\varepsilon\gamma^2}{(2n+\gamma)(2n+\gamma+2)}\right)P_n^{(\gamma)}(x) - \frac{4n^2(n+\gamma)^2}{(2n+\gamma)^2(2n+\gamma+1)(2n+\gamma-1)}P_{n-1}^{(\gamma)}(x), \quad (77)$$

$$k_n^{(\gamma)} = 2^{2n+\gamma+1}(n!)^2 \frac{(\Gamma(n+\gamma+1))^2}{(2n+\gamma+1)(\Gamma(2n+\gamma+1))^2}, \quad (78)$$

$$P_n^{(\gamma)} = \frac{\varepsilon^n 2^n n!}{\Gamma(\gamma+2n+1)} \sum_{m=0}^n \binom{n}{m} \frac{\Gamma(\gamma+n+m+1)}{m!} \left(\frac{\varepsilon x - 1}{2}\right)^m. \quad (79)$$

Now $P_n(x, M)$ is written in the basis of monic polynomials $P_j^{(\gamma-1)}(x)$ orthogonal with respect to \hat{c}_0 which is the linear functional associated to the measure $(1+\varepsilon x)^{\gamma-1}dx$ on the support $] -1, 1[$.

$$P_n(x, M) = \sum_{j=0}^n \theta_{j,n} P_j^{(\gamma-1)}(x), \quad \theta_{n,n} = 1.$$

Property 3.4 *The explicit expressions of $P_n(x, M)$ in the basis of monic Jacobi polynomials $P_j^{(\gamma-1)}(x)$, of the square norm $k_n^{(0)}$ of $P_n(x, M)$, and of σ_n are given by*

$$P_n(x, M) = P_n^{(\gamma-1)}(x) - \frac{Mn!2^n\varepsilon^n}{2\gamma+nM(n-1+\gamma)} \frac{\Gamma(n+\gamma)}{\Gamma(2n+\gamma)} \sum_{j=0}^{n-1} \frac{\varepsilon^j}{2^j} \frac{(\gamma+j)_{j+1}}{j!} P_j^{(\gamma-1)}(x). \quad (80)$$

$$k_n^{(0)} = \left(\frac{\Gamma(n+\gamma)}{\Gamma(2n+\gamma)}\right)^2 \frac{2^{2n+\gamma}(n!)^2}{2n+\gamma} \frac{2\gamma+(n+1)M(n+\gamma)}{2\gamma+nM(n-1+\gamma)}, \quad (81)$$

$$\sigma_n = \frac{\varepsilon n}{2n+\gamma+1} \left(\frac{\gamma}{2n+\gamma} - \frac{2M(\gamma+n)}{2\gamma+(n+1)M(n+\gamma)} \right) \quad (82)$$

$$= \frac{\varepsilon n(\gamma 2^\gamma + M(n+\gamma)(4n-\gamma(n-1)))}{(2n+\gamma+1)(2n+\gamma)(2\gamma+(n+1)M(n+\gamma))}. \quad (83)$$

Proof. We use the following orthogonality relations of $P_n(x, M)$

$$\begin{aligned} c_0(P_i^{(\gamma-1)}(x)P_n(x, M)) &= 0 \quad \text{for } i = 0, \dots, n-1 \\ &= \sum_{j=0}^n \theta_{j,n} \left(\hat{c}_0(P_i^{(\gamma-1)}(x)P_j^{(\gamma-1)}(x)) + MP_i^{(\gamma-1)}(\xi)P_j^{(\gamma-1)}(\xi) \right) \\ &= \frac{\theta_{i,n}2^{2i+\gamma}(i!)^2(\Gamma(i+\gamma))^2}{(2i+\gamma)(\Gamma(2i+\gamma))^2} + M\varepsilon^i 2^i i! \frac{\Gamma(i+\gamma)}{\Gamma(2i+\gamma)} \sum_{j=0}^n \theta_{j,n} \varepsilon^j 2^j j! \frac{\Gamma(j+\gamma)}{\Gamma(2j+\gamma)} \end{aligned}$$

Thus

$$\theta_{i,n} = -\frac{M\varepsilon^i(2i+\gamma)\Gamma(2i+\gamma)}{2^{i+\gamma}i!\Gamma(i+\gamma)} \sum_{j=0}^n \theta_{j,n} \varepsilon^j 2^j j! \frac{\Gamma(j+\gamma)}{\Gamma(2j+\gamma)}.$$

Let us set $K = \sum_{j=0}^n \theta_{j,n} \varepsilon^j 2^j j! \frac{\Gamma(j+\gamma)}{\Gamma(2j+\gamma)}$.

$$K = -M \sum_{j=0}^{n-1} (2j+\gamma)2^{-\gamma}K + \varepsilon^n 2^n n! \frac{\Gamma(n+\gamma)}{\Gamma(2n+\gamma)}$$

by using the expressions of the $\theta_{j,n}$'s in the definition of K . Hence

$$K = \frac{\varepsilon^n 2^n n!}{1 + M 2^{-\gamma} n(\gamma + n - 1)} \frac{\Gamma(n + \gamma)}{\Gamma(2n + \gamma)}$$

and

$$\begin{aligned} \theta_{j,n} &= -\varepsilon^{n+j} M \frac{2^{n-j} n!}{j!} \frac{(\gamma + j)_{j+1}}{2^\gamma + nM(n-1+\gamma)} \frac{\Gamma(n + \gamma)}{\Gamma(2n + \gamma)}, \quad j = 0, \dots, n-1, \\ \theta_{n,n} &= 1. \end{aligned} \quad (84)$$

Hence Relation (80) holds.

The square norm $k_n^{(0)}$ of $P_n(x, M)$ is given by

$$\begin{aligned} k_n^{(0)} &= c_0((P_n(x, M))^2) \\ &= \hat{c}_0((P_n(x, M))^2) + M(P_n(\varepsilon, M))^2 \\ &= \sum_{j=0}^n \theta_{j,n}^2 k_j^{(\gamma-1)} + M \left(\sum_{j=0}^n \theta_{j,n} P_j^{(\gamma-1)}(\varepsilon) \right)^2 \\ &= \left(\frac{\Gamma(n + \gamma)}{\Gamma(2n + \gamma)} \right)^2 \frac{2^{2n+\gamma} (n!)^2}{2n + \gamma} \frac{2^\gamma + (n+1)M(n + \gamma)}{2^\gamma + nM(n-1+\gamma)} \end{aligned}$$

by using the expressions of the $\theta_{j,n}$'s, of $k_j^{(\gamma-1)}$ and the value of $P_j^{(\gamma-1)}(\varepsilon)$. Hence (81) holds. σ_n will be obtained from

$$\frac{n}{n+1} \frac{P'_{n+1}(x, M) - (n+1)P_n^{(\gamma)}(x)}{P'_n(x, M)}$$

with the ratio of the coefficients of x^{n-1} in the numerator and the denominator.

The coefficient of x^{n-1} in $P'_{n+1}(x, M)$ is

$$\varepsilon n(n+1) \left(-1 + \frac{2n+1}{2n+\gamma+1} - \frac{2M}{2^\gamma + (n+1)M(n+\gamma)} \frac{\gamma+n}{\gamma+2n+1} \right).$$

The coefficient of x^{n-1} in $(n+1)P_n^{(\gamma)}(x)$ is

$$\varepsilon n(n+1) \left(-1 + \frac{2n}{2n+\gamma} \right).$$

Therefore (82) holds. \square

Like in the Laguerre case (b) a much more complicated way consists to use the following orthogonality relations

$$c_0((1 - \varepsilon x)^i P_n(x, M)) = 0, \quad i = 0, \dots, n-1. \quad (85)$$

The main interest of this kind of proof is to give explicitly $\hat{c}_0((1 - \varepsilon x)^i P_j^{(\gamma-1)})$ for $i \geq j$, and as a consequence a new identity implying a sum of products of binomial coefficients and Pochhammer coefficients. The expression of $\hat{c}_0((1 - \varepsilon x)^i P_j^{(\gamma-1)})$ for $i \geq j$ is given in the following Lemma. A Corollary will give the new identity.

Lemma 3.5 For $i \geq j$, $\forall j \in \mathbb{N}$,

$$\hat{c}_0((1 - \varepsilon x)^i P_j^{(\gamma-1)}(x)) = \frac{(-\varepsilon)^j 2^{i+j+\gamma} i! j!}{(\gamma + j)_{i+1}} \frac{\Gamma(j + \gamma)}{\Gamma(2j + \gamma)} \binom{i}{j}. \quad (86)$$

Proof.

Note that if $i = j$, the right hand side of (86) exactly is $(-\varepsilon)^j k_j^{(\gamma-1)}$. Moreover if $j = 0$, the right hand side of (86) is equal to $i! \frac{2^{\gamma+i}}{(\gamma)_{i+1}}$. It exactly is the value of $\hat{c}_0((1 - \varepsilon x)^i)$ which can be proved by recurrence very simply.

First of all we write the three term recurrence relation satisfied by the polynomials $P_j^{(\gamma-1)}$ as

$$\begin{aligned} P_{j+1}^{(\gamma-1)}(x) &= -\varepsilon(1 - \varepsilon x - \frac{(2j + \gamma - 1)(2j + \gamma + 1) - (\gamma - 1)^2}{(2j + \gamma - 1)(2j + \gamma + 1)})P_j^{(\gamma-1)}(x) \\ &\quad - \frac{4j^2(j + \gamma - 1)^2}{(2j + \gamma - 1)^2(2j + \gamma)(2j + \gamma - 2)}P_{j-1}^{(\gamma-1)}(x). \end{aligned} \quad (87)$$

We begin to prove that Lemma 3.5 holds for $i = j + 1$. (87) is multiplied by $(1 - \varepsilon x)^j$ and \hat{c}_0 is applied to the result. Since Lemma 3.5 holds for $i = 1$ and $j = 0$, we assume that it holds for $j = 0, \dots, r - 1$ and $i = j + 1$, and we prove that it holds for $j = r$ and $i = r + 1$. We have

$$\begin{aligned} \hat{c}_0((1 - \varepsilon x)^{r+1}P_r^{(\gamma-1)}(x)) &= \frac{(2r + \gamma - 1)(2r + \gamma + 1) - (\gamma - 1)^2}{(2r + \gamma - 1)(2r + \gamma + 1)}(-\varepsilon)^r k_r^{(\gamma-1)} \\ &\quad - \frac{\varepsilon 4r^2(r + \gamma - 1)^2}{(2r + \gamma - 1)^2(2r + \gamma)(2r + \gamma - 2)} \frac{(-\varepsilon)^{r-1} 2^{2r-1+\gamma} r! (r-1)!}{(\gamma + r - 1)_{r+1}} \times \\ &\quad \frac{\Gamma(r-1+\gamma)}{\Gamma(2r-2+\gamma)} \binom{r}{r-1} \\ &= \frac{(-\varepsilon)^r 2^{2r+1+\gamma} (r+1)! r!}{(\gamma + r)_{r+1}} \frac{\Gamma(r+\gamma)}{\Gamma(2r+\gamma)} \binom{r+1}{r}. \end{aligned}$$

Hence Lemma 3.5 holds for $i = r + 1$ and $j = r$.

Now we assume that Lemma 3.5 holds $\forall j \in \mathbb{N}$ and $i = j, \dots, j + r - 1$. We prove it for $i = j + r$, $\forall j \in \mathbb{N}$. (87) is multiplied by $(1 - \varepsilon x)^{j+r-1}$ and \hat{c}_0 is applied to the result. $\hat{c}_0((1 - \varepsilon x)^{r+j}P_j^{(\gamma-1)}(x))$ is expressed as a function of $\hat{c}_0((1 - \varepsilon x)^{r+j-1}P_{j+1}^{(\gamma-1)}(x))$, $\hat{c}_0((1 - \varepsilon x)^{r+j-1}P_j^{(\gamma-1)}(x))$ and $\hat{c}_0((1 - \varepsilon x)^{r+j-1}P_{j-1}^{(\gamma-1)}(x))$. By using the assumption of recurrence, we obtain

$$\hat{c}_0((1 - \varepsilon x)^{r+j}P_j^{(\gamma-1)}(x)) = \frac{(-\varepsilon)^j 2^{2r+j+\gamma} (r+j)! j!}{(\gamma + j)_{r+j+1}} \frac{\Gamma(j+\gamma)}{\Gamma(2j+\gamma)} \binom{r+j}{j}.$$

Hence Lemma 3.5 holds. \square

Corollary 3.6 For $i \geq j$,

$$\sum_{m=0}^j (-1)^m \binom{j}{m} \binom{i+m}{i} (\gamma + i + m + 1)_{j-m} (\gamma + j)_m = (-1)^j \binom{i}{j} (\gamma)_j. \quad (88)$$

Proof.

For $i \geq j$, by using (79) we have

$$\begin{aligned} \hat{c}_0((1 - \varepsilon x)^i P_j^{(\gamma-1)}) &= \frac{\varepsilon^j 2^j j!}{\Gamma(2j+\gamma)} \sum_{m=0}^j (-1)^m \binom{j}{m} \frac{\Gamma(j+m+\gamma)}{2^m m!} \hat{c}_0((1 - \varepsilon x)^{i+m}) \\ &= \frac{\varepsilon^j 2^{i+j+\gamma}}{(\gamma)_{i+j+1}} \frac{\Gamma(j+\gamma)}{\Gamma(2j+\gamma)} \sum_{m=0}^j (-1)^m \binom{j}{m} \binom{i+m}{i} \times \\ &\quad (\gamma + i + m + 1)_{j-m} (\gamma + j)_m. \end{aligned}$$

by using (86) for $j = 0$. At last Lemma 3.5 implies that (88) holds. \square

From Lemma 3.5 it remains to compute the θ_j 's. This part also is very technical. Since we already have this result, it is not necessary to give it.

Finally we have the following theorem concerning the Markov-Bernstein inequalities.

Theorem 3.7 *The following Markov-Bernstein inequality holds:*

$$c_1((p')^2) \leq \frac{1}{\mu_{1,n}} c_0(p^2), \quad \forall p \in \mathcal{P}_n$$

where $\mu_{1,n}$ is the smallest zero of the polynomials $A_n(\lambda)$ satisfying the following three term recurrence relation

$$A_n(\lambda) = (\lambda + B_n)A_{n-1}(\lambda) - C_n A_{n-2}(\lambda), \quad \forall n \geq 2. \quad (89)$$

where

$$B_n = -\frac{2^\gamma + (n+1)M(n+\gamma)}{2^\gamma + nM(n-1+\gamma)} \frac{2}{(2n+\gamma)(2n+\gamma-1)} - \frac{(\gamma 2^\gamma + M(n-1+\gamma)(4(n-1)-\gamma(n-2)))^2}{2(2^\gamma + nM(n-1+\gamma))(2^\gamma + (n-1)M(n-2+\gamma))} \times \frac{1}{(2n+\gamma-1)(2n+\gamma-2)(n-1+\gamma)^2}, \quad (90)$$

$$C_n = \frac{(\gamma 2^\gamma + M(n-1+\gamma)(4(n-1)-\gamma(n-2)))^2}{(2^\gamma + (n-1)M(n-2+\gamma))^2(n+\gamma-1)^2(2n+\gamma-1)(2n+\gamma-2)^2(2n+\gamma-3)}. \quad (91)$$

$A_0(\lambda) = 1$ and

$$A_1(\lambda) = \lambda - \frac{2(\gamma+1)^2}{\gamma+2} \frac{2^\gamma + 2M(1+\gamma)}{2^\gamma + M\gamma}. \quad (92)$$

Proof.

All these relations are obtained by replacing $k_{n-1}^{(0)}$, $k_n^{(0)}$, $k_{n-1}^{(1)}$, $k_{n-2}^{(1)}$ and σ_{n-1} in (2) by their expressions given in the relations (81), (78) and (83).

\square

Property 3.8

$$\lim_{n \rightarrow \infty} \mu_{1,n} = 0.$$

Proof.

We use Blumenthal Theorem 1.5. $\lim_{n \rightarrow \infty} B_n = 0$ and $\lim_{n \rightarrow \infty} C_n = 0$. Therefore $\sigma = 0$ and $\tau = 0$. Hence the property holds by using Remark 1.6. \square

3.3 Jacobi case (d)

The measure associated to c_0 is the Jacobi measure $(1-x)^\alpha(1+x)^\beta dx$ with $\alpha > -1$ and $\beta > -1$, and the one associated to c_1 is $\frac{(1-x)^{\alpha+1}(1+x)^{\beta+1}}{|x-\xi|} dx + M\delta(\xi)$ with $|\xi| \geq 1$ and $M \geq 0$. Therefore the monic polynomials $P_n(x)$, orthogonal with respect to c_0 , are the monic Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. Also here and for the same reasons as in the Laguerre case (c) we will denote by $c^{(\alpha,\beta)}$ the linear functional c_0 . For more convenience $|x-\xi|$ will be written as $\varepsilon(x-\xi)$ with $\varepsilon = 1$ if $\xi \leq -1$, and $\varepsilon = -1$ if $\xi \geq 1$. Let us denote by $T_n(x, \xi, M)$, $n \geq 0$, the monic polynomials orthogonal with respect to c_1 . Moreover we will denote by \tilde{c}_1 the linear functional associated to

the measure $\frac{(1-x)^{\alpha+1}(1+x)^{\beta+1}}{\varepsilon(x-\xi)}dx$ and therefore, we have $c^{(\alpha+1,\beta+1)}(.) = \tilde{c}_1(\varepsilon(x-\xi).)$.

Relation (1) becomes

$$T_n(x, \xi, M) = P_n^{(\alpha+1,\beta+1)}(x) - \sigma_n P_{n-1}^{(\alpha+1,\beta+1)}(x). \quad (93)$$

Like in the Laguerre case (c) we have a property giving a relation between $T_{n+1}(t, \xi, M)$ and the reproducing kernel $N_n^{(\alpha+1,\beta+1)}(x, t)$.

Property 3.9 *The monic polynomial $T_{n+1}(t, \xi, M)$, orthogonal with respect to c_1 , satisfies the following relation*

$$T_{n+1}(t, \xi, M) = -\frac{k_n^{\alpha+1}}{c_1(P_n^{(\alpha+1,\beta+1)}(x))} \left(1 - (t - \xi)c_1(N_n^{(\alpha+1,\beta+1)}(x, t))\right) \quad (94)$$

where the linear functional c_1 acts on the variable x and $N_n^{(\alpha+1,\beta+1)}(x, t)$ is the reproducing kernel

$$N_n^{(\alpha+1,\beta+1)}(x, t) = \sum_{j=0}^n \frac{P_j^{(\alpha+1,\beta+1)}(x)P_j^{(\alpha+1,\beta+1)}(t)}{k_j^{(\alpha+1,\beta+1)}}. \quad (95)$$

Proof.

$T_{n+1}(x, \xi, M)$ is written in the basis $\{1, \{\varepsilon(x - \xi)P_j^{(\alpha+1,\beta+1)}(x)\}_{j=0}^n\}$. Thus

$$T_{n+1}(x, \xi, M) = T_{n+1}(\xi, \xi, M) + \sum_{j=0}^n \theta_{j,n} \varepsilon(x - \xi) P_j^{(\alpha+1,\beta+1)}(x), \quad \theta_{n,n} = 1. \quad (96)$$

The sequel of the proof is the same as in Property 2.15. In the Jacobi case (d) we find

$$\theta_{i,n} = -\frac{T_{n+1}(\xi, \xi, M)}{k_i^{(\alpha+1,\beta+1)}} c_1(P_i^{(\alpha+1,\beta+1)}(x)), \quad i = 0, \dots, n, \quad (97)$$

and since $\theta_{n,n} = 1$ we have

$$T_{n+1}(\xi, \xi, M) = -\frac{k_n^{(\alpha+1,\beta+1)}}{c_1(P_n^{(\alpha+1,\beta+1)}(x))}. \quad (98)$$

□

The property giving \hat{k}_n and σ_n also has a similar proof as the one in Property 2.16.

Property 3.10 *The square norm \hat{k}_n of the polynomials $T_n(x, \xi, M)$ and σ_n are given by*

$$\begin{aligned} \hat{k}_0 &= c_1(1), \\ \hat{k}_n &= -k_{n-1}^{(\alpha+1,\beta+1)} \frac{c_1(P_n^{(\alpha+1,\beta+1)}(x))}{c_1(P_{n-1}^{(\alpha+1,\beta+1)}(x))}, \quad \forall n \geq 1, \end{aligned} \quad (99)$$

$$\sigma_n = \frac{c_1(P_n^{(\alpha+1,\beta+1)}(x))}{c_1(P_{n-1}^{(\alpha+1,\beta+1)}(x))}, \quad \forall n \geq 1. \quad (100)$$

At last we have the following theorem concerning the Markov-Bernstein inequalities.

Theorem 3.11 *The following Markov-Bernstein inequality holds:*

$$c_1((p')^2) \leq \frac{1}{\mu_{1,n}} c_0(p^2), \quad \forall p \in \mathcal{P}_n$$

where $\mu_{1,n}$ is the smallest zero of the polynomials $A_n(\lambda)$ satisfying the following three term recurrence relation

$$A_n(\lambda) = (\lambda + B_n)A_{n-1}(\lambda) - C_n A_{n-2}(\lambda), \quad \forall n \geq 2. \quad (101)$$

where

$$\begin{aligned} B_n &= \frac{4(n-1)(n+\alpha)(n+\beta)}{n(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)} \frac{c_1(P_{n-2}^{(\alpha+1,\beta+1)}(x))}{c_1(P_{n-1}^{(\alpha+1,\beta+1)}(x))} \\ &\quad + \frac{1}{(n-1)(n+\alpha+\beta)} \frac{c_1(P_{n-1}^{(\alpha+1,\beta+1)}(x))}{c_1(P_{n-2}^{(\alpha+1,\beta+1)}(x))}, \end{aligned} \quad (102)$$

$$\begin{aligned} C_n &= \frac{(n-2)(n+\alpha-1)(n+\beta-1)}{4(n-1)^2(n+\alpha+\beta)(2n+\alpha+\beta-1)(2n+\alpha+\beta-2)^2(2n+\alpha+\beta-3)} \times \\ &\quad \frac{c_1(P_{n-1}^{(\alpha+1,\beta+1)}(x))c_1(P_{n-3}^{(\alpha+1,\beta+1)}(x))}{(c_1(P_{n-2}^{(\alpha+1,\beta+1)}(x)))^2}. \end{aligned} \quad (103)$$

$A_0(\lambda) = 1$ and

$$A_1(\lambda) = \lambda - \frac{k_1^{(\alpha+1,\beta+1)}}{c_1(1)} = \lambda - \frac{2^{\alpha+\beta+3}\Gamma(\alpha+2)\Gamma(\beta+2)}{(\alpha+\beta+2)_2\Gamma(\alpha+\beta+3)c_1(1)}. \quad (104)$$

Proof.

By using (99), (100) and (16) we have

$$\begin{aligned} B_n &= -\frac{k_n^{(\alpha,\beta)}}{n^2 k_{n-1}^{(1)}} - \frac{\sigma_{n-1}^2}{(n-1)^2} \frac{k_{n-1}^{(\alpha,\beta)}}{k_{n-1}^{(1)}} \\ &= \frac{4(n-1)(n+\alpha)(n+\beta)}{n(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)} \frac{c_1(P_{n-2}^{(\alpha+1,\beta+1)}(x))}{c_1(P_{n-1}^{(\alpha+1,\beta+1)}(x))} \\ &\quad + \frac{1}{(n-1)(n+\alpha+\beta)} \frac{c_1(P_{n-1}^{(\alpha+1,\beta+1)}(x))}{c_1(P_{n-2}^{(\alpha+1,\beta+1)}(x))}, \\ C_n &= \frac{\sigma_{n-1}^2 (k_{n-1}^{(\alpha,\beta)})^2}{(n-1)^4 k_{n-1}^{(1)} k_{n-2}^{(1)}} \\ &= \frac{(n-2)(n+\alpha-1)(n+\beta-1)}{4(n-1)^2(n+\alpha+\beta)(2n+\alpha+\beta-1)(2n+\alpha+\beta-2)^2(2n+\alpha+\beta-3)} \times \\ &\quad \frac{c_1(P_{n-1}^{(\alpha+1,\beta+1)}(x))c_1(P_{n-3}^{(\alpha+1,\beta+1)}(x))}{(c_1(P_{n-2}^{(\alpha+1,\beta+1)}(x)))^2}. \end{aligned}$$

□

The Jacobi function of the second kind $Q_n^{(\alpha+1,\beta+1)}(\xi)$ can be used to replace the terms $c_1(P_n^{(\alpha+1,\beta+1)}(x))$ in B_n and C_n given in Theorem 3.11. $Q_n^{(\alpha,\beta)}(\xi)$ is defined as follows (see Szegő [17] Relations

(4.61.4) and (4.61.5) page 74) for $\xi \in \mathbb{C} \setminus [-1, 1]$:

$$Q_n^{(\alpha, \beta)}(\xi) = \frac{1}{2}(\xi - 1)^{-\alpha}(\xi + 1)^{-\beta} \int_{-1}^1 (1-x)^\alpha (1+x)^\beta \frac{\tilde{P}_n^{(\alpha, \beta)}(x)}{\xi - x} dx \quad (105)$$

$$= 2^{n+\alpha+\beta} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)} \frac{F(n+\alpha+1, n+1; 2n+\alpha+\beta+2; \frac{2}{1-\xi})}{(\xi-1)^{n+\alpha+1}(\xi+1)^\beta} \quad (106)$$

where $\tilde{P}_n^{(\alpha, \beta)}(x)$ is the non-monic Jacobi polynomial ($\tilde{P}_n^{(\alpha, \beta)}(x) = \frac{(n+\alpha+\beta+1)_n}{2^n n!} P_n^{(\alpha, \beta)}(x)$) and F is the hypergeometric function $F(a, b; c; z) = \sum_{\nu=0}^{\infty} \frac{(a)_\nu (b)_\nu}{(c)_\nu \nu!} z^\nu$.
Now the following property can be proved.

Property 3.12

$$\lim_{n \rightarrow \infty} \mu_{1,n} = 0.$$

Proof.

First we prove this result when $|\xi| > 1$. Then we will use the asymptotic behavior of $Q_n^{(\alpha+1, \beta+1)}(\xi)$ and of $P_n^{(\alpha+1, \beta+1)}(\xi)$ when n tends to infinity. We have (see Szegő [17] Relation (8.71.19) page 225)

$$(\xi - 1)^{\alpha+1}(\xi + 1)^{\beta+1} Q_n^{(\alpha+1, \beta+1)}(\xi) \simeq \frac{1}{\sqrt{n}} (\xi - \sqrt{\xi^2 - 1})^{n+1} \Phi(\xi)$$

where Φ is independent of n , and (see Szegő [17] Theorem 8.21.7 page 196)

$$\tilde{P}_n^{(\alpha+1, \beta+1)}(\xi) \simeq \frac{(\sqrt{\xi-1} + \sqrt{\xi+1})^{\alpha+\beta+2}}{(\xi-1)^{(\alpha+1)/2}(\xi+1)^{(\beta+1)/2}} \frac{(\xi + \sqrt{\xi^2 - 1})^{n+1/2}}{\sqrt{2\pi n} \sqrt{\xi^2 - 1}}.$$

Therefore

$$\begin{aligned} c_1(P_n^{(\alpha+1, \beta+1)}(x)) &= \frac{2^n n!}{(n+\alpha+\beta+3)_n} (\tilde{c}_1(\tilde{P}_n^{(\alpha+1, \beta+1)}(x)) + M \tilde{P}_n^{(\alpha+1, \beta+1)}(\xi)) \\ &= \frac{2^n n!}{(n+\alpha+\beta+3)_n} (2\varepsilon(\xi-1)^{\alpha+1}(\xi+1)^{\beta+1} Q_n^{(\alpha+1, \beta+1)}(\xi) \\ &\quad + M \tilde{P}_n^{(\alpha+1, \beta+1)}(\xi)). \end{aligned}$$

For ξ fixed, $|\xi| > 1$, we have $|\xi - \sqrt{\xi^2 - 1}| < 1$ and $\lim_{n \rightarrow \infty} |\xi - \sqrt{\xi^2 - 1}| = 0$. Therefore, if $M > 0$,

$$c_1(P_n^{(\alpha+1, \beta+1)}(x)) \simeq \frac{2^n n! (\xi + \sqrt{\xi^2 - 1})^{n+1/2}}{(n+\alpha+\beta+3)_n \sqrt{n}} \left(\frac{M (\sqrt{\xi-1} + \sqrt{\xi+1})^{\alpha+\beta+2}}{(\xi-1)^{\alpha/2}(\xi+1)^{\beta/2} \sqrt{2\pi} (\xi^2 - 1)} \right)$$

and if $M = 0$,

$$c_1(P_n^{(\alpha+1, \beta+1)}(x)) \simeq \frac{\varepsilon 2^{n+1} n! (\xi - \sqrt{\xi^2 - 1})^{n+1}}{(n+\alpha+\beta+3)_n \sqrt{n}} \Phi(\xi).$$

Hence, if $M > 0$, we have

$$\frac{c_1(P_{n-1}^{(\alpha+1, \beta+1)}(x))}{c_1(P_{n-2}^{(\alpha+1, \beta+1)}(x))} \simeq \frac{2\sqrt{(n-1)(n-2)}(n+\alpha+\beta+1)}{(2n+\alpha+\beta-1)_2} (\xi + \sqrt{\xi^2 - 1}).$$

If $M = 0$, we have

$$\frac{c_1(P_{n-1}^{(\alpha+1, \beta+1)}(x))}{c_1(P_{n-2}^{(\alpha+1, \beta+1)}(x))} \simeq \frac{2\sqrt{(n-1)(n-2)}(n+\alpha+\beta+1)}{(2n+\alpha+\beta-1)_2} (\xi - \sqrt{\xi^2 - 1}).$$

In the two cases we have $\lim_{n \rightarrow \infty} B_n = \lim_{n \rightarrow \infty} C_n = 0$. Therefore $\sigma = \tau = 0$ and the property holds by using Remark 1.6.

When $|\xi| = 1$, $\tilde{c}_1 = c^{(\alpha, \beta+1)}$ if $\xi = 1$ and $\tilde{c}_1 = c^{(\alpha+1, \beta)}$ if $\xi = -1$.
If $\xi = 1$, by using recursively (19) for $n, n-1, \dots, 1$ we obtain

$$\tilde{c}_1(P_n^{(\alpha+1, \beta+1)}(x)) = c^{(\alpha, \beta+1)}(P_n^{(\alpha+1, \beta+1)}(x)) = \frac{2^n n! (\beta+2)_n}{(\alpha+\beta+3)_{2n}} k_0^{(\alpha, \beta+1)}.$$

Then

$$\begin{aligned} c_1(P_n^{(\alpha+1, \beta+1)}(x)) &= \frac{2^{n+\alpha+\beta+2} n! (\beta+2)_n \Gamma(\alpha+1) \Gamma(\beta+2)}{(\alpha+\beta+3)_{2n} \Gamma(\alpha+\beta+3)} + M P_n^{(\alpha+1, \beta+1)}(1) \\ &= \frac{2^n (\alpha+2)_n}{(n+\alpha+\beta+3)_n} \left(M + 2^{\alpha+\beta+2} \frac{n!}{(\alpha+2)_n} \frac{(\beta+2)_n}{(\alpha+\beta+3)_n} \frac{\Gamma(\alpha+1) \Gamma(\beta+2)}{\Gamma(\alpha+\beta+3)} \right). \end{aligned}$$

Since $\alpha+1 > 0$ and $\beta+2 > 1$ the following quantities tend to 0 when n tends to ∞ :

$$\begin{aligned} \frac{n!}{(\alpha+2)_n} &= \prod_{j=1}^n \frac{1}{1 + \frac{\alpha+1}{j}}, \\ \frac{(\beta+2)_n}{(\alpha+\beta+3)_n} &= \prod_{j=1}^n \frac{1}{1 + \frac{\alpha+1}{\beta+1+j}}. \end{aligned}$$

Therefore, if $M > 0$,

$$c_1(P_n^{(\alpha+1, \beta+1)}(x)) \simeq M \frac{2^n (\alpha+2)_n}{(n+\alpha+\beta+3)_n}$$

and if $M = 0$,

$$c_1(P_n^{(\alpha+1, \beta+1)}(x)) = \frac{2^{n+\alpha+\beta+2} n! (\beta+2)_n \Gamma(\alpha+1) \Gamma(\beta+2)}{(\alpha+\beta+3)_{2n} \Gamma(\alpha+\beta+3)}.$$

Hence, if $M > 0$, we have

$$\frac{c_1(P_{n-1}^{(\alpha+1, \beta+1)}(x))}{c_1(P_{n-2}^{(\alpha+1, \beta+1)}(x))} \simeq \frac{2(n+\alpha)(n+\alpha+\beta+1)}{(2n+\alpha+\beta-1)_2}.$$

If $M = 0$, we have

$$\frac{c_1(P_{n-1}^{(\alpha+1, \beta+1)}(x))}{c_1(P_{n-2}^{(\alpha+1, \beta+1)}(x))} \simeq \frac{2(n+\beta)}{(2n+\alpha+\beta-1)_2}.$$

Again, in the two cases we have $\lim_{n \rightarrow \infty} B_n = \lim_{n \rightarrow \infty} C_n = 0$. Therefore $\sigma = \tau = 0$ and the property holds by using Remark 1.6.

If $\xi = -1$, by using (18) all the sequel is analogous. \square

4 Conclusion

The seven kinds of three term recurrence relation have been given. If $A_n(\lambda)$ is a polynomial obtained for one of them, the inverse of the square root of its smallest zero is the Markov-Bernstein constant linked to the corresponding coherent pair of measures.

Five kinds of three term recurrence relation are totally explicit. The last two relations are provided by means of integrals: $c_1(L_j^{(\alpha+1)}(x))$ in the Laguerre case (c) and $c_1(P_j^{(\alpha+1, \beta+1)}(x))$ in the Jacobi case (d) which can possibly be replaced by functions of the second kind.

In [4] and [6] classical numerical methods, in particular the Newton method and the qd algorithm, were used to obtain explicit upper and lower bounds of Markov-Bernstein constants. Here these methods have only been used successfully in the Laguerre case (b). Even in the case of the very simple three term recurrence relation (31) obtained in the Laguerre case (a) these methods do not provide an explicit result. But we have proved in the Laguerre case (a), the Laguerre case (b) when $\xi = 0$ and all the Jacobi cases that $\lim_{n \rightarrow \infty} \mu_{1,n} = 0$. Nevertheless these three term recurrence relations are the prerequisite in any advanced study of Markov-Bernstein constants linked to coherent pairs of measures. In particular the numerical value of $\mu_{1,n}$ can be computed from this three term recurrence relation in order to use it in some applications.

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